ON GLEASON PARTS FOR ALGEBRAS OF ANALYTIC FUNCTIONS AND MEASURES ORTHOGONAL TO THESE ALGEBRAS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.948.3:517.948.42

MATHEMATICS

R. E. VAL' SKII

ON GLEASON PARTS FOR ALGEBRAS OF ANALYTIC FUNCTIONS AND MEASURES ORTHOGONAL TO THESE ALGEBRAS

(Presented by Academician V. I. Smirnov on December 6, 1965)

In the present note we consider a variant of the problem of uniform approximation of functions continuous on a set and analytic at its interior points by rational fractions. We assume that for separate parts of the original set such an approximation is possible, and seek conditions on the character of the "gluing" of these parts under which approximation is possible on the whole set. It turns out that approximation "as a whole" will be possible if the separate parts do not adjoin one another too closely.

The main part of the proof consists in decomposing measures defined on the original set and orthogonal to rational functions into a sum of measures concentrated on its subsets. This decomposition is stronger than is needed for obtaining the approximation theorem mentioned above, and is of independent interest. A similar decomposition was considered earlier in the works of Bishop ([1,2]); for general Dirichlet algebras such a decomposition was constructed in the work of Glicksberg and Wermer ([3]). The results obtained are applied to the study of one interesting class of sets.

In the note the following notation will be used: X is a compact set in the complex plane; ∂X is the boundary of X; U is the interior of X; $U_1, \dots, U_n, \dots$ are the components of U; $C(X)$ is the algebra of all continuous functions on X with the usual norm; $R(X)$ is the subalgebra of $C(X)$ consisting of functions representable as the limit of a sequence of rational fractions with poles outside X, converging uniformly on X; $A(X)$ is the algebra of functions continuous on X and analytic at the interior points of X; $R^{\perp}(X)$ is the space of all finite complex-valued Borel measures μ, defined on ∂X and orthogonal to $R(X)$ (i.e. $\int_{\partial X} f\, d\mu = 0,\ f \in R(X)$); $\|\mu\| = \int_{\partial X} |d\mu|$. The (Cauchy) potential F_μ of a measure $\mu \in R^{\perp}(X)$ is defined as follows:

$$F_\mu(x) = \frac{1}{2\pi i} \int_{\partial X} \frac{d\mu(\xi)}{\xi - x} \quad (x \in U).$$

$K(z,\delta)=\{w: \ |w-z|<\delta\}$; $\overline{E}$ is the closure of the set E. By the symbol $f|_{E_0}$ we shall denote the restriction to the set E_0 of a function f defined on the set E, $E_0\subset E$; $\perp$ is the sign of mutual singularity of measures (two measures are called mutually singular if each is concentrated on a set of measure zero with respect to the other).

Let $x_1, x_2 \in X$; we shall write $x_1 \sim x_2$ if

$$\|x_1-x_2\|_X=\sup_{\substack{f\in R(X)\\ \|f\|\le 1}}|f(x_1)-f(x_2)|<2.$$

This relation was introduced by Gleason $(^4)$ for arbitrary function algebras. Note that always $\|x_1-x_2\|_X\le 2$. It can be shown $(^4)$ that the relation $\sim$ is an equivalence relation (i.e. reflexive, symmetric, and transitive). An equivalence class in the sense of $\sim$ is called a **part**. Each component U_i is contained entirely in one of the parts $(^4)$.

Let us pose the following question: can every measure $\mu\in R^{\perp}(X)$ be decomposed into a sum of mutually singular measures $\mu_i\in R^{\perp}(\overline{U}_i)$?

Theorem 1. *Let X be a plane compact set; let $U_1,\ldots,U_n,\ldots$ be the components of the interior of X; and let $R(\partial X)=C(\partial X)$. In order that every measure $\mu\in R^{\perp}(X)$ can be decomposed into a series convergent in variation*

$$\mu=\sum_i \mu_i,$$

where 1) μ_i is concentrated on ∂U_i, $\mu_i\in R^{\perp}(\overline{U}_i)$, 2) $\mu_i\perp\mu_j$ for $i\ne j$, it is necessary and sufficient that U_i and U_j not belong to the same part when $i\ne j$.

From general theorems of functional analysis and Theorem 1 there immediately follows

Theorem 2. *Let X be a plane compact set; let $U_1,\ldots,U_n,\ldots$ be the components of the interior of X; suppose U_i, U_j do not belong to one part for $i\ne j$; and $R(\partial X)=C(\partial X)$. If $f\in C(X)$, $f|_{\overline{U}_i}\in R(\overline{U}_i)$, $i=1,\ldots$, then $f\in R(X)$. In particular, if $A(\overline{U}_i)=R(\overline{U}_i)$, $i=1,\ldots$, then $A(X)=R(X)$.*

Proof of Theorem 1 (sufficiency). Let $\mu\in R^{\perp}(X)$. Using the definition of a part, construct a sequence $\{f_n: f_n\in R(X), \|f_n\|\le 1\}$ such that

$$f_n(x)\to 1 \quad \left(x\in\bigcup_{j<i}U_j\right), \qquad f_n(x)\to 0 \quad \left(x\in\bigcup_{j\ge i}U_j\right).$$

Define the measure ν_i as the weak limit of the measures $f_n\mu$. Then $\|\nu_i\|\le\|\mu\|$ and

$$F_{\nu_i}(x)=\begin{cases}F_\mu(x), & x\in\bigcup\limits_{j<i}U_j,\\ 0, & x\in\bigcup\limits_{j\ge i}U_j.\end{cases}$$

Put $\mu_i = \nu_{i+1} - \nu_i$. Then $\mu_k \in R^{\perp}(\overline{U}_k)$. From comparison of Cauchy potentials it follows that

$$\sum_k \mu_k = \mu.$$

Lemma 1. *Let X_1, X_2 be compact sets in the complex plane, $R(X_1) = C(X_1)$. Let $f \in C(X_1 \cup X_2)$, $f|_{X_2} \in R(X_2)$. Then $f \in R(X_1 \cup X_2)$.*

Corollary. μ_k is concentrated on ∂U_k, $\mu_k \in R^{\perp}(\overline{U}_k)$.

The mutual singularity of the measures μ_i and μ_j is established by

Lemma 2. *Suppose U_i and U_j do not belong to the same part, $\alpha \in R^{\perp}(\overline{U}_i)$, $\beta \in R^{\perp}(\overline{U}_j)$. Then $\alpha \perp \beta$.*

Proof. From the definition of a part it follows that there exists a sequence $\{g_n\}$ such that $g_n \in R(X)$, $\|g_n\| \le 1$, $g_n(x) \to 1$ $(x \in U_i)$, $g_n(x) \to -1$ $(x \in U_j)$. Then for any $\alpha' \in R^{\perp}(\overline{U}_i)$, $\beta' \in R^{\perp}(\overline{U}_j)$ we have

$$\alpha' - \beta' = \lim_{n\to\infty} g_n(\alpha' + \beta'), \qquad \alpha' + \beta' = \lim_{n\to\infty} g_n(\alpha' - \beta')$$

(in the weak sense), whence $\|\alpha' - \beta'\| = \|\alpha' + \beta'\|$. In particular,

$$\|f\alpha - \beta\| = \|f\alpha + \beta\| \qquad (f \in R(\overline{U}_i)).$$

One can obtain such an equality for every $f \in R(\partial X) = C(\partial X)$.

Let $\alpha = h\beta + \gamma$ be the Radon-Nikodym decomposition of the measure α with respect to the measure β (here $h \in L'(\beta)$, $\gamma \perp \beta$). Choosing a bounded sequence $\{f_n,\ f_n \in R(\partial X)\}$ such that $f_{nh} \to |h|$ (with respect to the measure β), we obtain $h = 0$ (almost everywhere with respect to the measure), and this means precisely that $\alpha \perp \beta$. In particular, $\mu_i \perp \mu_j$. The theorem is proved.

We now consider the parts of compact sets of the following special form. Let l_1, l_2 be rectifiable Jordan curves bounding domains U_1, U_2, with $U_1 \cap U_2 = \emptyset$, $l_1 \cap l_2 = \emptyset$. Let $X = \overline{U}_1 \cup \overline{U}_2$. We shall say that a compact set X of this form belongs to the class K.

The following theorem holds (by mes_1 we shall denote the natural measure on a rectifiable curve):

Theorem 3. 1) *Let X be a compact set of class K, bounded by the curves l_1, l_2.*

a) *If $\mathrm{mes}_1(l_1 \cap l_2) > 0$, then U_1 and U_2 lie in one part.*

b) If $\mathrm{mes}_1(l_1 \cap l_2) = 0$ and there exists a sufficiently smooth curve l such that U_1 lies inside l, and U_2 outside it, then U_1 and U_2 lie in different parts.

2) There exists a compact set X_0 of class K for which $\mathrm{mes}_1(l_1 \cap l_2) > 0$, and almost all points of $l_1 \cap l_2$ belong to one part together with $U_1 \cup U_2$.

Remark. The part constructed in 2) has no analytic structure ([4],[5]).

Proof of 1a). Define the measures dz_1, dz_2 as follows:
$dz_i(E) = \int_{E\cap l_i} dz$ ($i = 1, 2;$ E is a Borel set). Since $\mathrm{mes}_1(l_1 \cap l_2) > 0$, it follows that dz_1 is not $\perp$ dz_2. But $dz_i \in R^{\perp}(\overline{U_i})$ ($i = 1, 2$), and from Lemma 2 it follows that U_1 and U_2 lie in one part.

1b) From the definition of equivalence it follows that if $\widetilde{X} \supset X$ and U_1, U_2 lie in different parts for $\widetilde{X}$, then U_1 and U_2 lie in different parts for X. Therefore one may assume that X is already bounded by sufficiently smooth curves.

Using estimates of analytic capacity on "good" curves ([6],[7]) and the theorem of S. N. Mergelyan ([8]), we obtain that $A(X) = R(X)$. With the aid of Rudin' s theorem ([9]) one constructs a sequence $\{f_n,\ f_n \in A(X) = R(X),\ \|f_n\| \le 1\}$ such that $f_n(x) \to 1$ $(x \in U_1)$, $f_n(x) \to -1$ $(x \in U_2)$. Our assertion now follows from the definition of equivalence.

One can approach this question in another way as well. Let $\mu \in R^{\perp}(X)$. In ([11]) it is shown that there exists a measure μ', defined on l_1, such that $F_\mu = F_{\mu'}$ (in U_1). Then, for l_1 sufficiently smooth, the potential F_μ belongs to V. I. Smirnov' s class E_δ in U_1 for all $\delta < 1$ (for the definition of the class E_δ, see ([10])). Since $F_\mu = 0$ outside X, it follows from the basic lemma of I. I. Privalov ([10]) that F_μ has almost everywhere on l_1 summable angular boundary values (coinciding with the derivative, in the sense of Radon—Nikodym, of the measure μ with respect to the measure dz_1). If now l_1 belongs to the class C, then, by the theorem of V. I. Smirnov ([10]), $F_\mu \in E_1$ in U_1. In exactly the same way $F_\mu \in E_1$ in U_2. Bishop' s arguments ([2]) now show that $\mu \in A^{\perp}(X)$. Hence $A^{\perp}(X) = R^{\perp}(X)$ and $A(X) = R(X)$.

2) Put

$$X_0 = \overline{K(0,1)} \setminus \bigcup_{i=1}^{\infty} K(z_i, r_i),$$

where $\{z_i\}$ is a countable set dense on $[-1, 1]$. If $x_1, x_2 \in X_0$, then

$$\|x_1 - x_2\|_{X_0} = \lim_{n\to\infty} \|x_1 - x_2\|_{X_n},$$

where

$$X_n = \overline{K(0,1)} \setminus \bigcup_{i=1}^{n} K(z_i, r_i).$$

Moreover,

$$\|x_1 - x_2\|_{X_{n+1}} \le \|x_1 - x_2\|_{X_n} + \frac{4r_{i+1}}{\min(|x_1 - z_{i+1}|,\ |x_2 - z_{i+1}|) - r_{i+1}}.$$

If the r_i are chosen in accordance with this estimate, then X_0 will have the required property.

In conclusion I express my gratitude to my scientific adviser V. P. Khavin for his help in this work.

Leningrad State University
named after A. A. Zhdanov

Received
25 X 1965

REFERENCES

1. E. Bishop, Duke Math. J., 25, No. 2 (1958).
2. E. Bishop, Duke Math. J., 27, No. 3 (1960).
3. J. Wermer, I. Glicksberg, Duke Math. J., 30, No. 4 (1963).
4. A. M. Glicksberg, Collection of Translations. Mathematics, 5, 2 (1961).
5. G. Stolzenberg, J. Math. and Mech., 12, No. 1 (1963).
6. L. D. Ivanov, Uspekhi Mat. Nauk, 18, No. 4 (1963).
7. V. P. Khavin, S. Ya. Khavinson, Dokl. Akad. Nauk SSSR, 138, 14 (1961).
8. V. I. Smirnov, N. A. Lebedev, *Constructive Theory of Functions of a Complex Variable*, Moscow–Leningrad, 1964.
9. W. Rudin, Proc. Am. Math. Soc., 7, 808 (1956).
10. I. I. Privalov, *Boundary Properties of Analytic Functions*, 1950.
11. V. P. Khavin, Dokl. Akad. Nauk SSSR, 151, No. 3 (1963).