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Abstract

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ON BOUNDARY CONDITIONS FOR THE NAVIER–STOKES EQUATIONS IN A RAREFIED GAS

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1. In the kinetic approach to the formulation of gas-dynamic problems, various ways of passing to the macroscopic level of description are possible. For an unambiguous choice, additional considerations, criteria, and principles are required that are not contained in the Boltzmann description of the gas. We shall consider the question of boundary conditions in a monatomic gas, with the equations and the form of representation of the distribution function f in terms of gas-dynamic quantities already chosen. We shall regard the law of interaction of gas atoms with the surface as arbitrary, without tying ourselves to the scheme of specular-diffuse reflection. Taking as a basis the principle of combining the physical and mathematical closedness of the problem formulation, we arrive at the necessity of coordinating the boundary transform $\widetilde{T}$ with the representation f. The form and character of this connection, the class of admissible transforms, and a new version of boundary conditions for the Navier–Stokes (N.–S.) equations are the subject of this note.

2. Let

$$f(\mathbf{u}) = n(2h)^{3/2}\omega(\mathbf{v})\left[1 + \frac{1}{2}a_{ij}^{(2)}H_{ij}^{(2)}(\mathbf{v}) + \frac{1}{10}a_{ijj}^{(3)}H_{ijj}^{(3)}(\mathbf{v})\right], \qquad (1)$$

where $\omega(\mathbf{v}) = (2\pi)^{-3/2}\exp(-v^2/2)$, $\mathbf{v} = \sqrt{2h}(\mathbf{u} - \mathbf{U})$, the coefficients $a^{(2)}, a^{(3)}$ are small, and $H^{(l)}$ are Hermite polynomials [1].

The N.–S. equations describe the change of mass, momentum, and energy inside the domain. For the physical closedness of the problem at the boundary, it is necessary to take conditions describing the fluxes of all these quantities. From the relation [2]

$$u_n f(\mathbf{u})\big|_{u_n>0} = \iiint_{u_{1n}<0} |u_{1n}|\,\widetilde{T}(\mathbf{u},\mathbf{u}_1)f(\mathbf{u}_1)\,d\mathbf{u}_1 \qquad (2)$$

with normalized $\widetilde{T}$ we obtain $U_n = 0$,

$$\frac{a_{i3}+\delta_{i3}}{2} + \frac{1+\delta_{i3}}{\sqrt{2\pi}}\frac{a_{ijj}}{10} = I^{(1,0)}_{i,0} + \frac{a_{jk}}{2} I^{(1,2)}_{i,jk} + \frac{a_{kll}}{10} I^{(1,3)}_{i,kll}, \qquad i = 1,2,3, \tag{3}$$

$$\frac{1}{\sqrt{2\pi}}\left(1 + \frac{3}{2}a_{33}\right) + \frac{a_{3jj}}{2} = I^{(2,0)}_{ii,0} + \frac{a_{jk}}{2} I^{(2,2)}_{ii,jk} + \frac{a_{kll}}{10} I^{(2,3)}_{ii,kll}, \tag{4}$$

where index 3 corresponds to the normal direction,

$$I^{(l,m)} = \iiint_{v_{1n}<0} |v_{1n}|\,\omega(\mathbf{v}_1) H^{(m)}(\mathbf{v}_1) \iiint_{u_n>0} \widetilde{T}(\mathbf{u},\mathbf{u}_1) H^{(l)}(\mathbf{v})\,d\mathbf{u}\,d\mathbf{v}_1. \tag{5}$$

From the mathematical point of view, for the N.–S. equations it is sufficient to prescribe 4, not 5, conditions at the boundary. Bearing in mind that the approximate representation of f may impose certain requirements also on $\widetilde{T}$, we shall interpret the "extra" boundary condition as a condition on the transform that adapts it to the form of the distribution function.

3. To make $\tilde{T}$ consistent with f, put

$$\tilde{T} = \sqrt{2\pi}\,[v_n\omega(v) + \Delta\tilde{T}(\mathbf{v},\mathbf{v}_1;\mathbf{U},h)](2h)^{3/2}. \tag{6}$$

If $\Delta\tilde{T}$ is small, conditions (3), (4) are considerably simplified:

$$\frac{a_{i3}}{2} + \frac{1}{\sqrt{2\pi}}\frac{a_{ijj}}{10} = \Delta I^{(1)}_i(\mathbf{U},h), \quad i = 1,2, \tag{7}$$

$$\frac{a_{33}}{4} + \frac{1}{\sqrt{2\pi}}\frac{a_{3jj}}{5} = \Delta I^{(1)}_3(\mathbf{U},h), \qquad \frac{a_{33}}{\sqrt{2\pi}} + \frac{a_{3jj}}{2} = \Delta I^{(2)}_{ii}(\mathbf{U},h). \tag{8}$$

Here

$$\Delta I^{(l)}(\mathbf{U},h) = \sqrt{2h} \iiint_{v_{1n}<0} |v_{1n}|\,\omega(v_1) \iiint_{v_n>0} \Delta\tilde{T} H^{(l)}(\mathbf{v})\,d\mathbf{v}\,d\mathbf{v}_1. \tag{9}$$

Assuming the N.-S. equations to hold up to the boundary, it is natural to regard the slip velocity and the temperature jump as small. Then the near-Maxwellian transform becomes near-diffuse,

$$\tilde{T} = \frac{2h_w^2}{\pi}\,u_n \exp(-h^2_{wu})[1 + \varphi(\mathbf{w},\mathbf{w}_1)], \qquad \mathbf{w} = \sqrt{2h}\,\mathbf{u}, \tag{10}$$

so that

$$\Delta\tilde{T} = v_n\omega(v)\,[\varphi(\mathbf{v},\mathbf{v}_1) - \mathbf{v}\cdot\mathbf{V} - (h_s - 1)(v^2/2 - 2)],$$

$$h_s = h_w/h, \qquad \mathbf{V} = \sqrt{2h}\mathbf{U}, \tag{11}$$

and the boundary conditions (7), (8) are written in the form

$$\frac{a_{i3}}{2} + \frac{1}{\sqrt{2\pi}}\frac{a_{ijj}}{10} = -\frac{1}{\sqrt{2\pi}}V_i, \quad i = 1, 2, \tag{12}$$

$$\frac{1}{\sqrt{2\pi}}\frac{a_{3jj}}{5} = \varphi_3^{(1)} - \frac{1}{4}(h_s - 1), \qquad \frac{a_{3jj}}{2} = \varphi_{ii}^{(2)} - \frac{4}{\sqrt{2\pi}}(h_s - 1), \tag{13}$$

where $\varphi^{(l)}$ are determined by (9).

4. If the transform (10) is specified independently of f, then the boundary conditions for the normal momentum and energy (13) are incompatible. They can be made consistent by allowing φ to depend on the temperature jump. Putting

$$\varphi = \psi(\mathbf{w},\mathbf{w}_1) + (h_s - 1)\chi(\mathbf{w},\mathbf{w}_1), \tag{14}$$

we have the compatibility conditions

$$5\sqrt{2\pi}\,\psi_3^{(1)} = 2\psi_{ii}^{(2)}, \qquad 5\sqrt{2\pi}\,(\chi_3^{(1)} - 1/4) = 2(\chi_{ii}^{(2)} - 4/\sqrt{2\pi}). \tag{15}$$

To satisfy them it is enough to take

$$\psi = \frac{B(\mathbf{w}_1)}{10}H_{3jj}(\mathbf{w}), \qquad \chi = \frac{1}{2}[H_{ii}(\mathbf{w}) - 1] + \frac{C(\mathbf{w}_1)}{10}H_{3ij}(\mathbf{w}). \tag{16}$$

In this case equalities (13) reduce to a single relation. Introducing, for generality, into (14) a term proportional to the slip velocity, we arrive at boundary conditions for the N.-S. equations

$$V_3 = 0, \qquad V_i = -\sqrt{\pi/2}\,a_{i3} - a_{ijj}/10, \quad i = 1, 2, \tag{17}$$

$$a_{3jj} = \alpha + \beta V + \gamma(h_s - 1), \tag{18}$$

which differ somewhat from those adopted in the modern literature ([3],[4]). The condition relating the slip velocity to friction and the temperature creep proves to be universal, i.e., within the framework of the conserved

accuracy the same for all admissible transforms (the surface is isotropic). Condition (18), expressing the heat flux through the boundary in the form of a linear combination of the slip velocity and the temperature jump, combines the laws of transfer of normal momentum and energy. The terms α and βV in it are new, while the coefficient γ may be arbitrary. Thus, although matching on the whole plays a narrowing role, relation (18) turned out to be broader than the usual condition for the temperature jump.

5. In the case of specular-diffuse reflection, instead of (13) we have

$$h_s - 1 = \frac{\sigma - 2}{2\sigma} \frac{8}{5\sqrt{2\pi}} a_{3jj}, \qquad h_s - 1 = \frac{\sigma - 2}{2\sigma} \frac{\sqrt{2\pi}}{4} a_{3jj}. \tag{19}$$

The first of these conditions, which are incompatible for $\sigma \neq 0$, is in fact set aside in ([3], [4]). The error here is small, since the numerical coefficients in (19) are fairly close. Formal agreement can be achieved if, in the second relation, σ is replaced by α. However, distinguishing σ and α contradicts the model of specular-diffuse reflection, for the transfer of momentum and energy occurs in one process, and not in separate ones with different diffusivity fractions. Introducing σ and α at the macroscopic level as accommodation coefficients of momentum and energy, generally speaking unequal, is perfectly admissible; but this step should be regarded as going beyond the specular-diffuse scheme and as nullifying the usual derivation of the boundary conditions.

6. If the slip velocity and the temperature jump are not small, it is hardly meaningful to continue the N.–S. equations all the way to the wall. The boundary conditions are then better regarded as matching conditions at the outer boundary of a thin gas or thick adsorption layer, inside which similar distributions $\hat{T}/u_n$ and f are formed. Such an interpretation explains well, for example, the formulation of the problem for the Euler equations: $\Delta\hat{T} = 0$, conditions (7), (8) are satisfied identically, and physical closure is ensured. In the case of the N.–S. equations, the boundary conditions couple the external processes to the internal ones. The problem is closed by describing the phenomena inside the wall layer with a different representation f and boundary conditions on the surface itself. The introduction of such a layer significantly broadens the possibilities of the theory, making it possible to cover the nonuniform behavior of f near the boundary, especially at hypersonic speeds. Attempts in this direction ([5], [6]) should be continued and brought to a closed formulation of the problem. More general will be the expansion of f in finite subspaces of u ([7]), ensuring a uniform approach to the boundary without stratification of the space r.

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Note: Figure translations are in progress. See original paper for figures.