Periodic solutions of second-order nonlinear equations

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Abstract

The paper presents the conditions for the existence and provides an algorithm for constructing periodic solutions of the equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right),$$

the right-hand side of which is periodic in t. Bibliography: 6 items.

Full Text

Introduction

In 1967, A. M. Samoilenko [1, 2] proposed and investigated a numerical-analytical method for studying periodic solutions of the system:

$$\frac{dx}{dt} = f(t, x)$$

where $f(t, x)$ is a periodic function in t. In these works, the existence of periodic solutions was established, and an algorithm for their construction was provided. The present paper extends this method to systems of the form:

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{1.1}$$

where the function $f(t, x, y)$ is periodic in t with period T.

In Section 1, we prove the existence of periodic solutions for system (1.1). Section 2 provides a method for constructing these solutions, while Section 3 discusses the practical application of the results obtained in Section 1. We demonstrate that the periodic solution of (1.1) can be found as the limit of a sequence of periodic functions.

Section 1. Existence of Periodic Solutions

Consider the second-order differential equation (1.1). We assume that the function $f(t,x,y)$ is defined in the domain:

$$-\infty < t < \infty, \quad a \le x \le b, \quad c \le y \le d \tag{1.2}$$

and is periodic in t with period T. Furthermore, we assume f satisfies the Lipschitz condition:

$$|f(t,x,y) - f(t,x',y')| \le K_1|x - x'| + K_2|y - y'| \tag{1.3}$$

where K_1 and K_2 are positive constants.

Following the methodology in [1], we define the operator L as:

$$Lf(t) = \int_0^t [f(\tau) - \bar{f}]d\tau \tag{1.4}$$

where $\bar{f}$ is the average value of the function:

$$\bar{f} = \frac{1}{T}\int_0^T f(t)dt \tag{1.5}$$

We further define the iterated operator $L^2 f(t) = L(Lf)$. It was shown in [1] (Lemma 1) that the following estimate holds:

$$|Lf(t)| \le \alpha_1(t)|f|_0 \tag{1.7}$$

where $\alpha_1(t) = 2t(1 - t/T)$ for $0 \le t \le T$, and $|\cdot|_0$ denotes the maximum norm.

Theorem 1. Suppose the function $f(t,x,y)$ satisfies the Lipschitz condition (1.3) and is bounded by M in the domain (1.2). If the constants a,b,c,d,M,K_1,K_2 satisfy the conditions:

$$(A) \quad b - a \ge \dots, \quad d - c \ge \dots$$

$$(B) \quad \alpha_1(t)(K_1 + K_2) < 1$$

then there exists a sequence of functions $x_n(t,x_0)$ defined by the recurrence relation:

$$x_{n+1}(t,x_0) = x_0 + L^2 f(t, x_n(t,x_0), \dot{x}_n(t,x_0)) \tag{I}$$

which converges uniformly to a periodic function $x_\infty(t,x_0)$. This limit function satisfies the integro-differential equation:

$$x(t,x_0) = x_0 + L^2 f(t, x(t,x_0), \dot{x}(t,x_0)) \tag{III}$$

The proof proceeds by induction. For $m = 0$, we choose $x_0(t,x_0) = x_0$. From (I), we obtain:

$$|x_1(t,x_0) - x_0| \le \alpha_1(t)M$$

Given the constraints on the domain, we ensure that $x_m(t, x_0)$ remains within the interval $[a, b]$ and its derivative $\dot{x}_m(t, x_0)$ remains within $[c, d]$ for all m.

To prove convergence, we examine the difference between successive approximations:

$$|x_{m+1}(t) - x_m(t)| \leq |L^2[f(t, x_m, \dot{x}_m) - f(t, x_{m-1}, \dot{x}_{m-1})]| \quad (1.9)$$

Applying the Lipschitz condition and the properties of the operator L, we derive:

$$|x_{m+1}(t) - x_m(t)|_0 \leq Q_0 |x_m(t) - x_{m-1}(t)|_0 \quad (1.14)$$

where $Q_0 < 1$ is a constant derived from condition (B). This ensures the uniform convergence of the sequence $x_m(t)$ to $x_\infty(t)$ and $\dot{x}_m(t)$ to $\dot{x}_\infty(t)$.

Section 2. Relationship to the Original Differential Equation

The limit function $x_\infty(t, x_0)$ obtained in Section 1 is a solution to the original differential equation (1.1) if and only if the following condition is satisfied:

$$\Delta(x_0) = \frac{1}{T} \int_0^T f(t, x_\infty(t, x_0), \dot{x}_\infty(t, x_0))dt = 0 \quad (2.5)$$

This condition represents the "averaging" of the perturbation over one period. If there exists a value $x_0 = x^*$ such that $\Delta(x^*) = 0$, then $x_\infty(t, x^*)$ is a periodic solution of (1.1) with initial condition $x(0) = x^*$.

Theorem 3. If the function $\Delta(x_0)$ changes sign or vanishes at some point $x_0 \in [a, b]$, then the system (1.1) possesses a periodic solution. In practice, we approximate $\Delta(x_0)$ using the m-th iteration:

$$\Delta_m(x_0) = \frac{1}{T} \int_0^T f(t, x_m(t, x_0), \dot{x}_m(t, x_0))dt \quad (2.7)$$

The error of this approximation is bounded by:

$$|\Delta(x_0) - \Delta_m(x_0)| \leq \delta_m \quad (2.8)$$

where $\delta_m \to 0$ as $m \to \infty$.

Section 3. Numerical Application and Examples

The numerical-analytical method is particularly effective for systems with symmetries.

Theorem 5. If the function $f(t, x, y)$ satisfies the symmetry condition:

$$f(-t, x, -y) = -f(t, x, y)$$

then the system (1.1) has a periodic solution passing through $x_0 = 0$, provided the domain conditions are met. In this case, the approximating functions

$x_m(t,0)$ are odd, and the integral condition $\Delta_m(0) = 0$ is satisfied automatically for all m.

As an application, consider the motion of a mechanical system described by:

$$\frac{d^2\theta}{dt^2} + \nu^2\theta = F(t) \tag{3.3}$$

where $F(t)$ is a periodic forcing function. Using the proposed method, we can determine the stability regions and the amplitude of periodic oscillations by analyzing the zeros of the functional $\Delta(x_0)$.

References

1. Samoilenko, A. M., *Numerical-analytical method for investigating periodic solutions*, Ukrainian Mathematical Journal, Vol. XVII, No. 4, 1965.
2. Samoilenko, A. M., *On the periodic solutions of non-linear systems*, Ukrainian Mathematical Journal, Vol. XVIII, No. 2, 1966.
3. Bogolyubov, N. N., Mitropolsky, Y. A., *Asymptotic Methods in the Theory of Non-linear Oscillations*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.