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Abstract

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ON A LOWER ESTIMATE OF THE FREQUENCIES OF VIBRATIONS OF ELASTIC SYSTEMS

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The paper sets forth a method for estimating from below the vibration frequencies of a certain class of elastic systems, which we call connected systems. The idea of the method consists in replacing the given elastic system by a sequence of less rigid systems, and from this point of view it is close to ideas expressed in the works of a number of authors ([1,2]). In those works, if they are stated in the terms of mechanics, it is assumed that at least one elastic system is known which possesses a number of necessary properties and is certainly less rigid than the given one. In the present note, for the class of elastic systems considered by us, such an assumption is absent, since the indicated auxiliary system is constructed from the given one.

Let two elastic linear systems S and S_1 be given, occupying the domains D and D_1, for which $\{u(x)\}$ and $\{u^{(1)}(x_1)\}$ (x and x_1 are points of the domains D and D_1) are, respectively, the totalities of all possible displacements, and $I(u) \geqslant 0$, $I_1(u^{(1)})$ are their potential energies.

Suppose that between the points ξ and ξ_1 of two sets $E \subset D$ and $E_1 \subset D_1$ there has been established, in some manner, a one-to-one and mutually continuous correspondence. The system $S + S_1$, occupying the domain $D + D_1$, is called *connected* if every displacement equal to $u(x)$ for $x \in D$ and equal to $u^{(1)}(x_1)$ for $x_1 \in D_1$, with $u(\xi) = u^{(1)}(\xi_1)$, is its possible displacement, and there are no other possible displacements in the system; the potential energy of the system $S + S_1$ is equal to $I(u) + I_1(u^{(1)})$.

Introduce into consideration some sequence of functions of bounded variation $\{\bar{p}_i(x)\}$ $(i = 1, 2, ...)$ $(x \in D)$, subject to the conditions $d\bar{p}_i(x)$ for $x \in E$; every continuous function $f(x)$ satisfying the equalities

$$\int_E f(x)\, dp_i(x) = 0 \qquad (i = 1, 2, ...),$$

is equal to zero on E. Introduce also another sequence of functions $\{\bar{p}_i^{(1)}(x_1)\}$, defined in D_1 as follows: $d\bar{p}_i^{(1)}(x_1) = 0$ for $x_1 \in E_1$, $d\bar{p}_i^{(1)}(\xi_1) = d\bar{p}_i(\xi)$.

Denote, respectively, by $\bar{v}_i(x)$ and $\bar{v}_i^{(1)}(x_1)$ the displacements of the systems S and S_1 due to the loads $d\bar{p}_i(x)$ and $d\bar{p}_i^{(1)}(x_1)$, applied to the elements of their volumes, and put

$$\bar{a}_{ij} = \int_E \bar{v}_i \, d\bar{p}_j + \int_{E_1} \bar{v}_i^{(1)} \, d\bar{p}_j^{(1)}.$$

From the sequences $\{\bar{p}_i(x)\}$ and $\{\bar{p}_i^{(1)}(x_1)\}$ select subsequences consisting of all functions for which all the determinants $|\bar{a}_{ij}| \neq 0$ $(i, j = k_1, k_2, \ldots, k_n;\ n = 1, 2, \ldots)$*. The functions of these subsequences will be denoted by $p_i(x)$, $p_i^{(1)}(x_1)$, the corresponding displacements by $v_i(x)$, $v_i^{(1)}(x_1)$, and the corresponding constants by a_{ij}.

* The possibility of constructing such a subsequence follows from the nonnegativity of the potential energies of the systems S and S_1.

If the system S has an influence function $K(x, s)$, then the theorem proved by one of the authors ([3,4]) is valid.

Theorem. If $K(x, s)$ is a continuous kernel, then the influence function of the connected system $G(x, s)$ $(x, s \in D)$ is the limit, as $n \to \infty$, of a uniformly convergent sequence of kernels

$$G_n(x, s) = \frac{1}{A_n} \begin{vmatrix} K(x, s) & v_1(x) & \cdots & v_n(x) \\ v_1(s) & a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ v_n(s) & a_{n1} & \cdots & a_{nn} \end{vmatrix}. \tag{1}$$

Here $A_n = |\alpha_{ij}|_{i,j=1}^n$.

Let us show how the frequencies of oscillations of the connected system are estimated from below if at every point of the set E there is mass and outside this set there are no masses.

If $d\sigma(x)$ is the mass of an element of volume of the connected system ($d\sigma(x) > 0$ for $x \in E$, $d\sigma(x) = 0$ for $x \notin E$), then the squares of the frequencies of this system μ_i are equal to the characteristic numbers of the integral equation ([5])

$$\varphi(x) = \mu \int_E G(x, s) \varphi(s) \, d\sigma(s) \qquad (x \in E).$$

From the positivity, for $n < k$, of the kernels $G_n(x, s) - G_k(x, s)$ it follows that the characteristic numbers ν_i^n $(n = 1, 2, \ldots)$ of the equations

$$\varphi(x) = \nu \int G_n(x,s)\varphi(s)\,d\sigma(s) \qquad (x \in E) \tag{2}$$

for any i are subject to the inequalities *

$$\nu_i^{(n)} \leqslant \nu_i^{(k)} \leqslant \mu_i,$$

and since the kernels $G_n(x,s) - G(x,s)$ tend uniformly to zero as $n \to \infty$, then, as is known ([6]),

$$\lim_{n\to\infty} \nu_i^{(n)} = \mu_i.$$

In what follows we shall assume that the influence function of the system S is closed on the set E. This means that every continuous function $f(x)$ satisfying the condition

$$\int_E K(x,s) f(s)\,d\sigma(s) = 0,$$

is identically equal to zero on E. Under this assumption it is not difficult to verify that, in constructing the kernels $G_n(x,s)$, one may put

$$dp_i(x) = \varphi_i(x) d\sigma(x) \qquad (i = 1, 2, \ldots), \tag{3}$$

where $\{\varphi_i(x)\}$ is the sequence of all fundamental functions of the loaded integral equation

$$\varphi(x) = \int_E K(x,s)\varphi(s)\,d\sigma(s) \qquad (x \in E). \tag{4}$$

Theorem. If the kernel $K(x,s)$ is closed on E and the functions $p_i(x)$ are defined by the equalities (3), then every characteristic number of equation (2) either is a characteristic number of equation (4), or is a root of a polynomial whose degree is not greater than n.

Indeed, from the orthogonality of the fundamental functions $\varphi_i(x)$ and equality (1) it follows that for $j > n$ each characteristic number λ_j and each fundamental function $\varphi_j(x)$ of equation (4) will be a characteristic number and a fundamental function of equation (2). Therefore each fundamental function of equation (2), distinct from the listed

* The characteristic numbers of each kernel are arranged in order of increasing magnitude.

···by virtue of the closedness of the kernel $K(x, s)$ on E, will have the form

$$\varphi(x) = \sum_{i=1}^{n} c_i \varphi_i(x).$$

Substituting this value of $\varphi(x)$ into (2) and assuming

$$\int_E \varphi_i \varphi_j \, d\sigma = \delta_{ij},$$

after simple algebraic transformations we obtain that every root of the equation

$$\operatorname{Det} \left\| \lambda_i a_{ij} - \nu \left(a_{ij} - \delta_{ij}/\lambda_i \right) \right\|_{i,j=1}^{n} = 0 \tag{5}$$

will be a characteristic number of equation (2). Here

$$a_{ij} = \frac{\delta_{ij}}{\lambda_i} + \int_{E_1} v_i^{(1)} \varphi_j \, d\sigma.$$

If all roots of equation (5) and all numbers λ_j $(j > n)$ are arranged in increasing order, then the spectrum of characteristic numbers of the kernel $G_n(x, s)$ will be constructed; with its aid the frequencies of oscillations of the connected system are estimated.

Let us show how the frequencies of oscillations of a homogeneous square plate of constant thickness, clamped along the contour, are estimated. If a plate with cylindrical rigidity B is located in the square $0 \leq x, y \leq a$, then it may be assumed to be formed by the connection of two systems S and S_1, located in the same square, while E and E_1 coincide with D. The system S has potential energy

$$I(u) = \frac{B}{2} \int_0^a \int_0^a \left[\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 \right] dx \, dy,$$

where $u(x, y)$ are functions continuous together with their first partial derivatives, on which the integral $I(u)$ has meaning, and which are equal to zero on the contour.

The system S_1 has potential energy

$$I_1(u^{(1)}) = B \int_0^a \int_0^a \left(\frac{\partial^2 u^{(-1)}}{\partial x \, \partial y} \right)^2 dx \, dy,$$

where $u^{(1)}(x, y)$ are continuous functions on which the integral $I_1(u^1)$ has meaning, and which are equal to zero on the contour.

It can be shown that the influence function of the system S is a continuous kernel, closed in D. It is easy to see that all normalized fundamental functions and characteristic numbers of this kernel are equal to

$$\varphi_{ij}(x, y) = \sqrt{1/\rho}\, \psi_i(x)\psi_j(y); \qquad \lambda_{ij} = (\lambda_i + \lambda_j)B/\rho,$$

where ρ is the mass per unit area of the plate; $\psi_i(x)$ and λ_i are the normalized fundamental functions and eigenvalues of the boundary-value problem

$$\psi^{\mathrm{IV}} - \lambda\psi = 0; \qquad \psi(0) = \psi(a) = 0; \qquad \psi'(0) = \psi'(a) = 0.$$

Determination of the displacements of the system S from the required loads is performed simply, since these displacements are solutions of the differential equations

$$2B\, \partial^4 v_{ij}^{(1)}(x, y)/\partial x^2 \partial y^2 = \sqrt{1/\rho}\, \psi_i(x)\psi_j(y),$$

equal to zero on the boundary of the square.

In forming equation (5) we restricted ourselves to 6 functions $\varphi_{ij}(x, y)$ $(i+j \leq 4)$. The roots of this equation are

$$\nu_i = \alpha_i B/\rho a^4,$$

where $\alpha_1 = 1237$; $\alpha_2 = \alpha_3 = 5009$; $\alpha_4 = 10045$; $\alpha_5 = 16950$; $\alpha_6 = 16989$.

The numbers ν_i $(i = 1, \dots, 6)$ will be the first 6 characteristic numbers of the kernel $G_6(x, s)$, since $\min_{i+j>4} \lambda_{ij} > \nu_6$. The lower estimate of the frequencies has been obtained.

To estimate the frequencies from above, the Ritz method was applied. The computations were carried out using 6 coordinate functions $\varphi_{ij}(x, y)$ $(i + j \leq 4)$.

We give the final results. If ω_i are the vibration frequencies of the plate and $\beta_i = \omega_i a^2 \sqrt{\rho/B}$, then

$$35.20 \leq \beta_1 \leq 36.11; \quad 70.71 \leq \beta_2 = \beta_3 \leq 73.65; \quad 100.23 \leq \beta_4 \leq 108.50; \quad 130.19 \leq \beta_5 \leq 131.70; \quad 130.34 \leq \beta_6$$

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Note: Figure translations are in progress. See original paper for figures.