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Abstract

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ON FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE LAPLACE OPERATOR IN AN ARBITRARY DOMAIN AND ON AN ASYMPTOTIC ESTIMATE OF FUNDAMENTAL NUMBERS

(Presented by Academician A. N. Tikhonov on January 6, 1967)

Numerous works devoted to spectral questions in the theory of elliptic operators invariably attach to the study of the eigenfunctions and eigenvalues of these operators boundary conditions formulated either explicitly or in the form of specifying the domain of definition of the corresponding operator.

In the present work, apparently for the first time, complete orthonormal systems of functions corresponding to the simplest elliptic operator—the Laplace operator—are studied completely independently of boundary conditions.

Definition. We shall call a complete orthonormal system of functions $\{u_n(x)\}$ in an arbitrary N-dimensional domain g a **fundamental system of functions of the Laplace operator** *(elliptic operator L)* if each function $u_n(x)$ belongs in the open domain g to the class $C^{(2)}$ and, for some nonnegative number λ_n, satisfies in this domain the equation $\Delta u_n + \lambda_n u_n = 0$ ($Lu_n + \lambda_n u_n = 0$).

The numbers λ_n shall be called **fundamental numbers**.

Since every orthonormal system is at most countable, the fundamental numbers may be renumbered.

An example of a fundamental system of functions of a self-adjoint elliptic operator may be the system of eigenfunctions of this operator corresponding to any of the three homogeneous boundary-value problems.

Theorem 1. *For an arbitrary N-dimensional domain g and an arbitrary fundamental system of functions of the Laplace operator (or of a self-adjoint elliptic operator L of second order whose coefficients $a_{ij}(x)$ have bounded first derivatives), the sequence of fundamental numbers $\{\lambda_n\}$ is unbounded.*

We outline the proof of this theorem for the Laplace operator. Fix in the domain g some point y and a number R, smaller than the minimum distance of y from the boundary of the domain g, and put

$$f(x) = \begin{cases} r_{xy}^{7/4-N/2} - R^{7/4-N/2} + \left(\frac{N}{4} - \frac{7}{8}\right) R^{-N/2-1/4}(r_{xy}^2 - R^2), & \text{if } r_{xy} \le R, \\ 0, & \text{if } r_{xy} > R. \end{cases} \tag{1}$$

It is easy to verify that everywhere in the domain g, for $x \neq y$, the function $f(x)$ is continuous, has continuous first and piecewise continuous second derivatives. The Laplacian of this function everywhere for $x \neq y$ is equal to

$$\Delta f(x) = \begin{cases} N\left(\frac{N}{2} - \frac{7}{4}\right) R^{-N/2-1/4} - \left(\frac{N}{2} - \frac{7}{4}\right)\left(\frac{N}{2} - \frac{1}{4}\right) r_{xy}^{-N/2-1/4}, & \text{if } r_{xy} < R, \\ 0, & \text{if } r_{xy} > R. \end{cases} \tag{2}$$

Applying Green' s formula over the domain g to the two functions $f(x)$ and $u_n(x)$, and denoting the Fourier coefficients of the functions $f(x)$ and $\Delta f(x)$ with respect to the system $\{u_n(x)\}$ by f_n and $(\Delta f)_n$, respectively, we obtain

$$(\Delta f)_n = -\lambda_n f_n. \tag{3}$$

Since the function (1) belongs to the class $L_2(g)$, the series $\sum_{n=1}^{\infty} f_n^2$ converges. If the sequence $\{\lambda_n\}$ were bounded, then from relation (3) we would obtain that the series $\sum_{n=1}^{\infty} (\Delta f)_n^2$ also converges, but this is impossible, since the function (2) does not belong to the class $L_2(g)$.

For a self-adjoint elliptic operator whose coefficients $a_{ij}(x)$ have bounded first derivatives, the proof is carried out analogously: the same function (1) is taken as the basis.

A natural question arises: can the fundamental numbers λ_n have limit points on the numerical axis.

Theorem 2. *If g is an arbitrary bounded N-dimensional domain, and $\{u_n(x)\}$ is an arbitrary fundamental system of functions of the Laplace operator such that $u_n(x) \in W_2^1(g)$, then the fundamental numbers $\{\lambda_n\}$ have no limit points on the numerical axis (i.e., on any finite interval there lies only a finite number of fundamental numbers).*

Let us outline the proof of this theorem. Suppose that the fundamental numbers λ_n have a limit point λ. Consider the infinite sequence $\{\lambda_k\}$ of fundamental numbers lying in the interval $(\lambda - 1, \lambda + 1)$, and the corresponding sequence of fundamental functions $\{u_k(x)\}$.

Fix an arbitrary subdomain g' of the domain g, a point y in the subdomain g', and a number R smaller than the minimum distance between the boundaries of the domains g and g'. Then, writing Bessel' s inequality for the function

$$F(x) = \begin{cases} \Gamma(N/2+1)\pi^{-N/2}R^{-N}, & \text{for } r_{xy} < R, \\ 0, & \text{for } r_{xy} > R, \end{cases}$$

we are convinced of the uniform, with respect to $y \in g'$, convergence of the series

$$\sum_{k=1}^{\infty} u_k^2(y).$$

Consequently, the sequence $\{u_k(y)\}$ is infinitely small uniformly in the subdomain g'.

From this it is established that the sequence $\{u_k(y)\}$ is infinitely small uniformly in the entire domain, which contradicts the normalization of the functions $u_k(y)$ in the domain g. This final stage of the proof is far from simple and makes essential use of the formula established by us for expanding an arbitrary fundamental function in a Fourier series in spherical functions.

For the two-dimensional case, the indicated formula has the form

$$u_k(r,\varphi) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{2\pi} \frac{J_n(r\sqrt{\lambda_k})}{(\sqrt{\lambda_k}/2)^n} \left[\cos n\varphi \int_0^{2\pi} \frac{\partial^n u}{\partial r^n}(0,\psi)\cos n\psi \, d\psi + \sin n\varphi \int_0^{2\pi} \frac{\partial^n u}{\partial r^n}(0,\psi)\sin n\psi \, d\psi \right].$$

Here $\varepsilon_n = 1$ for $n \geq 1$; $\varepsilon_0 = 2$; r and φ are polar coordinates; the pole is located at an arbitrary point y of the domain g, with the radius r such that the disk of radius r centered at the point y is entirely contained in g.

It follows from Theorem 2 that *the fundamental numbers of the Laplace operator in an arbitrary domain can be renumbered in increasing order.*

In what follows we shall everywhere assume that all the conditions of Theorem 2 are satisfied and that the fundamental numbers are numbered in increasing order. In this case it follows from Theorem 1 that

$$\lim_{n\to\infty} \lambda_n = \infty. \tag{4}$$

Equality (4) makes it possible to apply to a completely arbitrary fundamental system of functions of the Laplace operator the apparatus developed in my works (1-3), since in those works only the completeness and orthonormality of the system under consideration, the mean-value formula (valid for any fundamental function), and the limit (4) are used. The following theorem gives an upper estimate of the fundamental numbers of the Laplace operator.

Theorem 3. *For an arbitrary bounded N-dimensional domain g and any fundamental system of functions of the Laplace operator satisfying requirement (4), there exists a positive constant B such that, for the fundamental numbers λ_n, the inequality*

$$\lambda_n \leqslant B n^{2/N} \tag{5}$$

holds.

The proof of this theorem is carried out according to the following scheme. In my work (1) a method is indicated for obtaining, in any strictly interior subdomain g', an asymptotic formula for eigenfunctions of the Laplace operator of the form

$$\sum_{k=1}^{n} u_k(x) u_k(y) = \left(\frac{\sqrt{\lambda_n}}{2\pi r_{xy}} \right)^{N/2} J_{N/2}\left(r_{xy}\sqrt{\lambda_n} \right) + O\left(\lambda_n^{(N-1-\varepsilon/2)} \right) \tag{6}$$

(here ε is any positive number). Although in the cited work eigenfunctions of specific boundary-value problems are discussed, in deriving formula (6) only the mean-value formula for the functions $u_n(x)$, the orthonormality of the system $\{u_n(x)\}$, and equality (4) are used. Thus formula (6) is valid for an arbitrary fundamental system of functions of the Laplace operator.

Putting $x = y$ in formula (6) and integrating the formula thus obtained over the subdomain g', we shall have

$$\sum_{k=1}^{n} \int_{g'} u_k^2(x)\, dx = \lambda_n^{N/2} \left[C + O(1) \right], \tag{7}$$

where

$$C = \frac{2^{-N} \pi^{-N/2} \operatorname{mes} g'}{\Gamma(N/2+1)} > 0.$$

If, in the left-hand side of (7), the integration were performed not over the subdomain g' but over the whole domain g (from which the indicated left-hand side could only increase), then the left-hand side of (7) would be equal to n. Thus inequality (5) is proved.

Suppose now that the fundamental system of functions of the Laplace operator possesses the following property: there exists a closed domain Ω lying inside g such that

$$\lim_{n\to\infty}\int_\Omega u_n^2(x)\,dx > 0.$$

We shall call this property the B**-property**. It is possible that an arbitrary fundamental system of functions of the Laplace operator possesses the B-property, but we have not yet succeeded in proving this.

Theorem 4. *If a fundamental system of functions of the Laplace operator possesses the B-property, then there exists a positive constant A such that, for the fundamental numbers λ_n, the inequality $An^{2/N} \leqslant \lambda_n$ holds.*

The theorems proved allow one to assert that all the results of my paper [2] and the results of my paper [3] concerning Riesz summability are valid for an arbitrary fundamental system of functions of the Laplace operator satisfying condition (4), while the results of paper [3] concerning summability by the Cesàro and Poisson-Abel methods are valid for a fundamental system of functions of the Laplace operator possessing the B-property.

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Note: Figure translations are in progress. See original paper for figures.