COHOMOLOGY OF LIE GROUPS

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Abstract

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COHOMOLOGY OF LIE GROUPS

WITH REAL COEFFICIENTS

1. By G we shall always denote a connected closed Lie subgroup of the group $GL(n, \mathbb{R})$ of nonsingular matrices of order n. By a **principal G-fibration** ξ we mean a triple (E, ξ, X), where E is a completely regular topological space, ξ is an action of G on E without fixed points (i.e., for every $g \in G$ there is given a homeomorphism $\xi(g) : E \to E$), and $X = E/G$ is the space of orbits. Let $\xi^{\#} : E^{\#} \to X$ be the real vector fibration with n-dimensional fiber and structure group G, induced by the principal fibration ξ. We shall call the fibration $\xi^{\#}$ **locally trivial** if for each fiber $F \subset E^{\#}$ there exists a continuous mapping $E^{\#} \to F$, identical on F and linear on each fiber. We shall call the principal fibration ξ locally trivial if the induced fibration is locally trivial. The usual condition of local triviality follows from ours and is equivalent to it if the space X is completely regular. In what follows, when speaking of a **principal G-fibration**, we shall always mean a locally trivial fibration whose base X is Hausdorff.

By a mapping $f : \xi_1 \to \xi_2$ of a fibration $\xi_1 = (E_1, \xi_1, X_1)$ into a fibration $\xi_2 = (E_2, \xi_2, X_2)$ we mean a mapping $f : E_1 \to E_2$ commuting with the action of the group. We note that there is no notion of an induced fibration in our category of principal G-fibrations.

2. Let A be an abelian topological group, and X a topological space. By $H^q_b(X; A)$ (when this cannot lead to misunderstanding, i.e., when the space X is not equipped with any group structure, we write simply $H^q(X; A)$) we denote the q-dimensional cohomology group of X with coefficients in the sheaf of germs of continuous mappings into A. We shall say that a class of q-dimensional cohomology of the group G with coefficients in A is given if for every principal G-fibration $\xi = (E, \xi, X)$ a class $\alpha(\xi) \in H^q(X; A)$ is defined, and moreover, if $f : \xi_1 \to \xi_2$ is a mapping of fibrations and $\varphi : X_1 \to X_2$ is the corresponding mapping of their bases, then $\varphi^* \alpha(\xi_2) = \alpha(\xi_1)$. If all classes α for a given group G form a set, then this set is a group with respect to the natural operation. This group is denoted by $H^q_{\text{alg}}(G; A)$.* We note that, by narrowing our category of principal fibrations to those whose base is a CW-complex in the case when the group A is discrete, we obtain the usual definition of the cohomology of a group. However, from the homotopy invariance of cohomology groups it follows that, in the case of a discrete group A, our definition gives the same groups $H^q_{\mathrm{alg}}(G; A)$ as the usual one.

The subject of the present note is the computation of the groups $H^q(G; \mathbb{R})$, where $\mathbb{R}$ is the (topological) group of real numbers. As is known,

* Analogously one can define the groups $H^q_{\mathrm{alg}}(G; M)$, where M is a topological group on which the group G acts; in particular, the cohomology groups of the group with coefficients in any of its representations are defined.

the cohomologies of any paracompact space with coefficients in the sheaf of germs of continuous real-valued functions are trivial in positive dimensions, and therefore the weakening, introduced above, of the restrictions on the base of the bundle is necessary. Intuitively, this may be explained by the fact that the trajectory space of a noncompact Lie group is, as a rule, bad from the topological point of view, so that it is natural to expect that the case when this space is good is, in a certain respect, trivial. But for a compact group the cohomologies that we compute turn out to be zero. It is interesting that for noncompact groups these cohomologies are always nontrivial.

3. The classical computation of the cohomology of groups consists in the computation of the cohomology of a classifying space. In our case the role of the classifying space is somewhat different, since cohomologies with coefficients in the sheaf of germs of real-valued functions are not homotopy invariants. Nevertheless, we need a certain analogue of a universal bundle.

Let I be some set. Consider the Tikhonov product

$$V_I = \prod_{i \in I} R^n_i,$$

where R^n_i $(i \in I)$ are copies of the n-dimensional real vector space. Obviously, V_I is a linear topological space; by $\mathcal{E}_I$ we denote the set of all n-frames of this space. The group $GL(n; \mathbf{R})$, and therefore also the group G, acts without fixed points on the space $\mathcal{E}_I$. By S_I we denote the trajectory space of the group G. It is easy to verify that in this way we have defined a principal G-bundle in the sense of § 1. We denote this bundle by ξ_I. A mapping $\eta : I_1 \to I_2$ induces a mapping of the bundle ξ_{I_1} into the bundle ξ_{I_2}.

Now let $\xi = (E, \xi, X)$ be a principal G-bundle. As is known, the space $E^{\#}$ of the induced vector bundle is obtained from the direct product $E \times \mathbf{R}^n$ by identifying $(a, \beta) = ((\xi(g)a, g^{-1}\beta)$. Thus, a mapping $E \times \mathbf{R}^n \to E^{\#}$ is defined; that is, for each point $a \in E$ a linear mapping f_a of the standard copy of $\mathbf{R}^n$ onto one of the layers of the bundle $\xi^{\#}$ is defined, and $f_{\xi(g)a}$ is a mapping onto the same layer as f_a, and

$$f_{\xi(g)a} = g \circ f_a.$$

Since the bundle $\xi^{\#}$ is locally trivial, for each $x \in X$ there exists a mapping $\varphi_x : E^{\#} \to \mathbf{R}^n_x$, linear on the layer $\mathbf{R}^n_x \subset E^{\#}$ and identical on $\mathbf{R}^n_x$. All these mappings together constitute a mapping

$$\varphi : E^{\#} \to \prod_{x \in X} \mathbf{R}^n_x = V_X,$$

which maps each layer of the bundle $\xi^{\#}$ linearly onto an n-dimensional plane in V_X. For each $a \in E$, the composition $\varphi \circ f_a$ is a linear nonsingular mapping of the standard copy of $\mathbf{R}^n$ into V_X, i.e., an n-frame in V_X. Thus, to each point of the space E we have assigned a point of the space $\mathcal{E}$, i.e., we have obtained a mapping $E \to \mathcal{E}$. This mapping is continuous and commutes with the action of the group G on E and on $\mathcal{E}$.

Thus, the principal bundles ξ_I are universal in the sense that every bundle is mapped into the bundle ξ_I for a set I of sufficiently large cardinality. To what extent different mappings $X \to S_I$ correspond in this case to different bundles with one and the same base X, we do not know. In any case, every element $a \in H^q_{alg}(G; A)$ is completely characterized by the elements

$$a(\xi_I) \in H^q_{top}(S_I; A)$$

for all sets I.

Example. Let $G = \mathbf{R} \subset GL(1; \mathbf{R})$ be the group of real numbers. Then $\mathcal{E}_I$ is the space of I-frames in the Tikhonov product of lines R_i, $i \in I$, i.e.,

$$\mathcal{E}_I = V_I \setminus 0.$$

The action of the group $\mathbf{R}$ on $\mathcal{E}_I$ is described by the formula

$$\xi(r)\{x_i,\ i \in I\} = \{e^r x_i,\ i \in I\},$$

where $x_i \in R_i,\ r \in \mathbf{R}$. The trajectory space S_I is an analogue of the sphere in the space V_I, but S_I does not lie in V_I and is not a regular space; moreover, all continuous functions on S_I are constants. A principal $\mathbf{R}$-bundle with base X may, by the known

in a manner that assigns an element of $H^1_{\mathrm{top}}(X; \mathbf{R})$. This correspondence is an element of $H^1_{\mathrm{alg}}(\mathbf{R}; \mathbf{R})$. This element is nonzero, since the foliation ξ_I, if the set I is infinite, is nontrivial. Consequently,

$$H^1_{\text{alg}}(\mathbf{R}; \mathbf{R}) \neq 0.$$

Secs. 4 and 5 contain the statements of several lemmas needed for the proof of the main theorem. A reader interested only in the statements should go directly to Sec. 6.

4. The following two topological properties of the spaces $\mathcal{E}_I$ are easily derived directly from the definition of the topology on them.

Proposition 1. *Let f be some continuous real-valued function on the space $\mathcal{E}_I$. There exists a subset $I' \subset I$ such that the difference $I \setminus I'$ is at most countable and the restriction of the function f to $\mathcal{E}_{I'} \subset \mathcal{E}_I$ is constant.*

Proposition 2. *Denote by $\mathbf{R}^*$ the sheaf of germs of continuous real-valued functions on $\mathcal{E}_I$ that are smooth on each leaf of the foliation ξ_I. Then $H^q(\mathcal{E}_I; \mathbf{R}^*) = 0$ for $q > 0$. (Smoothness is always understood to mean class C^∞.)*

In fact, we shall need the following strengthened version of Proposition 2.

Proposition $2'$. *Denote by $\widetilde{\mathbf{R}}$ the sheaf of germs of sections of the sheaf $\mathbf{R}^*$ over complete inverse images of neighborhoods of points of the space S_I (the base of this sheaf is the space S_I). Then $H^q(S_I; \widetilde{\mathbf{R}}) = 0$ for $q > 0$.*

5. We now proceed to compute the groups $H^q_{\text{alg}}(G; \mathbf{R})$. Introduce the following notation: $\mathbf{R}$ is the group of real numbers in the discrete topology; $\mathfrak{G}$ is the Lie algebra of the group G; $H^q(\mathfrak{G}; \mathbf{R})$ is the q-dimensional cohomology group of the Lie algebra with coefficients in the trivial representation, i.e., the cohomology of the complex of left-invariant forms on the group G.

Let $\xi = (E, \xi, X)$ be some principal G-foliation. We shall say that a differential form of order q is defined on this foliation if such a form (with smooth coefficients) is given on each leaf of this foliation and its coefficients depend continuously on the point of the base. The group of all such forms will be denoted by $\Omega^q(\xi)$. Obviously, the differential

$$d_q : \Omega^q(\xi) \to \Omega^{q+1}(\xi)$$

is defined. The cohomology of the complex $(\Omega^q(\xi), d_q)$ will be denoted by $h^q(\xi)$.

Proposition 3. *Let I be any set. There exists a cohomological spectral sequence of rings $\{E_2^{p,q}\}$ such that*

$$E_2^{p,q} = H^p(S_I; \mathbf{R}) \otimes H^q_{\text{top}}(G; \mathbf{R})$$

and E_∞ is associated with

$$h^* = \sum_q h^q(\xi_I).$$

Indeed, denote by $\widetilde{\mathbf{R}}_q$ the sheaf of germs of differential forms of order q on the foliation ξ_I (the base of this sheaf is S_I). From Proposition $2'$ it follows that the cohomology of S_I in all these sheaves is trivial. (Since the leaf of the foliation ξ_I is a group, all leaves can be parallelized simultaneously, and the sheaf $\widetilde{\mathbf{R}}_q$ is the sum of C_m^q sheaves $\widetilde{\mathbf{R}}^*$ from Proposition $2'$, where $m = \dim G$.) Consider the sequence

$$\mathbf{R} \to \widetilde{\mathbf{R}}_0 \xrightarrow{d} \widetilde{\mathbf{R}}_1 \xrightarrow{d} \cdots \xrightarrow{d} \widetilde{\mathbf{R}} \to 0$$

of sheaves over S_I. All these sheaves are acyclic, but the sequence is not exact, and therefore it leads not directly to the cohomology of S_I, but to the spectral sequence needed by us.

A map of the set I_1 into the set I_2 corresponds to a map of the corresponding spectral sequences. Therefore one can construct a "limiting" spectral sequence for which

$$E_2^{p,q} = H^p_{\mathrm{alg}}(G;\mathbf{R}) \otimes H^q_{\mathrm{top}}(G;\mathbf{R}).$$

This sequence converges to the groups $E_\infty^{p,q}$ associated with the groups

$$h^* = \sum h^q,$$

defined as follows. An element $a \in h^q$ is a function assigning to each

to the principal G-bundle $\xi = (E, \xi, X)$, an element $\alpha(\xi) \in h^q(\xi)$ is natural with respect to maps of bundles. But, as was already noted, to specify an element of $h^q(\xi)$ is the same as to specify a C_m^q-function on E; hence from Proposition 1 it follows that h^q is the homology of the complex of left-invariant forms on the group G, i.e. $h^q = H^q(\mathfrak{G};\mathbf{R})$. Thus, the following holds.

Theorem 1. *There is a (cohomological) spectral sequence of rings such that*

$$E_2^{p,q} = H^p_{lg}(G;\mathbf{R}) \otimes H^q_{top}(G,\mathbf{R})$$

and E_∞ is associated with

$$h^* = \sum H^q(\mathfrak{G};\mathbf{R}).$$

The homomorphism

$$H^q(\mathfrak{G};\mathbf{R}) \to E_\infty^{0,q} \to E_2^{0,q} = H^q(G;\mathbf{R})$$

coincides with the usual mapping of these groups.

In some cases (namely, when the map $H^q(\mathfrak{G};\mathbf{R}) \to H^q_{top}(G;\mathbf{R})$ is epimorphic) Theorem 1 already gives a definitive answer. The following theorem makes it possible to solve the problem completely.

Theorem 2. *Let m be the dimension of the group G, and k the dimension of its compact part. Then*

$$H^q_{alg}(G; \mathbf{R}) = 0$$

for $q > m - k$.

In particular, it follows from this theorem that if the group G is compact, then

$$H^q_{a\text{-}g}(G; \mathbf{R}) = 0$$

for $q > 0$. The same follows also from Theorem 1 (for a compact group the natural homomorphism $H^q(\mathfrak{G}; \mathbf{R}) \to H^q_{top}(G; \mathbf{R})$ is an isomorphism, see ([1]), Ch. 3). However, this is also clear directly.

The idea of the proof of Theorem 2 is that one considers the space $\hat{\mathcal{E}}_I$ of trajectories of a maximal compact subgroup $\hat{G}$ of the group G and the bundle $\hat{\mathcal{E}}_I \to S_I$, whose fibers are smooth manifolds diffeomorphic to E^{m-k}. The sequence of sheaves analogous to that considered in the proof of Proposition 3 is already exact, all the sheaves entering it are acyclic, and its length is equal to $m - k$. The assertion of the theorem follows from this.

6. From Theorems 1 and 2 one can obtain the groups $H^q_{alg}(G; \mathbf{R})$ for any group G. In the general case one obtains the following result.

Theorem 3. *The groups $H^q_{alg}(G; \mathbf{R})$ coincide with the cohomology groups of the complex of differential forms on the homogeneous space $G/\hat{G}$ that are invariant with respect to the action of the group G (here $\hat{G}$ is a maximal compact subgroup of the group G).*

In two special cases the formulation of this theorem can be made more concrete:

Corollary 1. *Let G be a semisimple Lie group, and $\bar{G}$ its compact form. Then*

$$H^q_{alg}(G; \mathbf{R}) = H^q_{top}(\bar{G}/\hat{C}; \mathbf{Q}) \otimes_Q \mathbf{R}.$$

Corollary 2. *Let G be a solvable Lie group. Then*

$$H^q_{alg}(G; \mathbf{R}) = H^q(\mathfrak{G}; \mathbf{R}),$$

where $\mathfrak{G}$ is the Lie algebra of the group G.

A detailed exposition of the results of this note will be included in our paper ([2]).

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References

[1] *Topology of Groups and Lie Algebras*, IL, 1962.
[2] I. M. Gelfand, D. B. Fuks, *Functional Analysis and Its Applications*, 1, No. 4 (1967).

Note: Figure translations are in progress. See original paper for figures.