METHODS FOR SOLVING GENERALIZED PROBLEMS OF SMOOTHING AND EXTRAPOLATION OF RANDOM FUNCTIONS

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Abstract

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D. B. YUDIN

METHODS FOR SOLVING GENERALIZED PROBLEMS OF SMOOTHING AND EXTRAPOLATION OF RANDOM FUNCTIONS

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In (^{1}) the conditions for the existence of a solution of generalized problems of smoothing and extrapolation of random functions were investigated, and a formal apparatus for the analysis of these problems was constructed. In the present note constructive methods are set forth for solving the problems formulated in (^{1}). The article uses the concepts and preserves the notation and terminology adopted in (^{1}).

Sec. 1. Let us formulate problem I of smoothing and extrapolation of random functions in terms of the Hilbert space H_r^n.

Problem I. It is required to choose a system of elements $h^\alpha \in G^\alpha$, $\alpha = 1, 2, \dots, n$, on which the quality index of the forecast $R(k^{\alpha\beta}) = \bar{R}(h^\alpha)$ of the random vector-function $\eta(t) = \{\eta^\alpha(t)\}$ at the point $t_0 + t_y$ attains its upper bound.

We associate with problem I problem I^σ of smoothing and extrapolation by the minimum of variance. Let us formulate it in terms of the space H_r^n.

Problem I^σ. It is required in each of the G^α to select an element h_σ^α on which the minimum $k^{\alpha\alpha}$ is attained.

Let us denote the correlation matrices of forecast errors corresponding to the forecast quality indices $R(k^{\alpha\beta})$ and $\sigma_\alpha^2 = k^{\alpha\alpha}$, $\alpha = 1, 2, \dots, n$, respectively, by $\|k_R^{\alpha\beta}\|$ and $\|k_\sigma^{\alpha\beta}\|$.

Theorem 1. *The correlation matrices of the errors of smoothing and extrapolation corresponding to the solutions of problems I and I^σ are related by the relation*

$$\|k_R^{\alpha\beta}\| = \|k_\sigma^{\alpha\beta}\| + \|k_P^{\alpha\beta}\|, \tag{1}$$

where $\|k_P^{\alpha\beta}\|$ is some nonnegative definite symmetric matrix—the matrix of artificial scattering.

Introduce the following notation. $\mathcal{P}^R_{\alpha\beta}(\tau)$, $\mathcal{P}^\sigma_{\alpha\beta}(\tau)$, $\alpha, \beta = 1, 2, \dots, n$, are the weight functions corresponding to the solutions of problems I and I^σ, respectively; $w(t_0, \tau) = w(\tau)$ is an arbitrary function belonging to $H^{(K)}(t_0, T)$, distinct from the identically zero function and satisfying the conditions

$$\int_0^T \psi_j^\alpha(t_0 - \tau) w(\tau)\, d\tau = 0, \qquad \alpha = 1, 2, \dots, n; \quad j = 1, 2, \dots, r.$$

Let, moreover,

$$a_{\alpha\beta} = \int_0^T \int_0^T k_{\xi^\alpha, \xi^\beta}(t_0 - \tau_1, t_0 - \tau_2) w(\tau_1) w(\tau_2)\, d\tau_1\, d\tau_2. \tag{2}$$

Define the matrix of numbers $\|\chi_{\alpha\beta}\|$, $\alpha, \beta = 1, 2, \dots, n$, by the equation

$$\|\chi_{\beta\mu}\|\,\|a_{\mu\nu}\|\,\|\chi_{\alpha\nu}\|^T = \|k_R^{\alpha\beta}\| - \|k_\sigma^{\alpha\beta}\| = \|k_P^{\alpha\beta}\|. \tag{3}$$

System (3) has a solution (more precisely, an infinite set of solutions), since there always exists a transformation with matrix $\|\chi_{\beta\mu}\|$ which transforms the positive definite quadratic form with matrix $\|a_{\mu\nu}\|$ into the nonnegative definite quadratic form with matrix $\|k_P^{\alpha\beta}\|$.

The matrices satisfying equation (3) are

$$\|x_{\alpha\beta}\| = \|c_{\alpha\mu}\|\|d_{\mu\nu}\|\|h_{\nu\lambda}\|\|c_{\lambda\beta}\|^{-1}. \tag{4}$$

Here $\|c_{\alpha\beta}\|$ is a nonsingular matrix which simultaneously reduces the matrix $\|a_{\alpha\beta}\|$ to normal form and the matrix $\|k_P^{\alpha\beta}\|$ to canonical form:

$$\|c_{\alpha\mu}\|\|a_{\mu\nu}\|\|c_{\alpha\nu}\|^T = E_n,$$

$$\|c_{\alpha\mu}\|\|k_P^{\mu\nu}\|\|c_{\alpha\nu}\| = \left\|\begin{matrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & g_n \end{matrix}\right\|,$$

$$\|h_{\nu\lambda}\| = \left\|\begin{matrix} \sqrt{g_1} & 0 & \cdots & 0 \\ 0 & \sqrt{g_2} & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \sqrt{g_n} \end{matrix}\right\|,$$

$\|d_{\mu\nu}\|$ is an arbitrary orthogonal matrix.

There is the following relation between the weight functions determining the solutions of Problems I and I^σ.

Theorem 2. *The weight functions on which the solutions of Problems I and I^σ are attained are connected by the formulas*

$$\mathcal{P}^R_{\alpha\beta} = \mathcal{P}^\sigma_{\alpha\beta} + x_{\alpha\beta} w(\tau), \qquad \alpha, \beta = 1, 2, \dots, n. \tag{5}$$

2.

Let us formulate Problem II of smoothing and extrapolation in terms of the spaces H^n and $H^{(K)}(t_0, T)$.

Problem II. It is required to choose a random vector $\zeta = \{\zeta^\alpha\} \subset L = L^n(t_0, T) \subset H^n$ (or, equivalently, a system of weight functions $\mathcal{P}_{\alpha\beta}(t_0, \tau) \in H^{(K)}(t_0, T)$, $\alpha, \beta = 1, 2, \dots, n$), for which the quality criterion of the prediction

$$R(m^\alpha, k^{\alpha\beta}) = \bar{R}(\zeta^\alpha) = R(\mathcal{P}_{\alpha\beta})$$

of the random vector-function $\eta(t) = \{\eta^\alpha(t)\}$ at the point $t_0 + t_y$ attains its upper bound.

Let us select in $L = L^n(t_0, T)$ the set of elements G_c of the form $\xi = \xi_0 + c$ $(c = \bar{\xi})$ and associate with Problem II the auxiliary Problems II^{σ_1} and II^{σ_2}.

Problem II^{σ_1}. It is required, for each α, $\alpha = 1, 2, \dots, n$, to find a random variable $\xi^\alpha \in G_0$ on which the minimum of

$$\mathrm{M}[\eta^\alpha(t_0 + t_y) - \xi^\alpha]^2$$

is attained.

Problem II^{σ_2}. It is required to find a random vector $\xi \in G_1$ on which the minimum of $\mathrm{M}[\xi]^2$ is attained.

Introduce the following notation: ξ^α_R, $\xi^\alpha_{\sigma_1}$, $\xi^\alpha_{\sigma_2}$ are random variables determining the solutions of Problems II, II^{σ_1}, and II^{σ_2}, respectively; m^α_R, $k^{\alpha\beta}_R$ are the first and second moments of the smoothing and extrapolation errors corresponding to the solution of Problem II;

$$c^\alpha_R = \mathrm{M}\xi^\alpha_R = \mathrm{M}\eta^\alpha - m^\alpha_R; \quad k^{\alpha\beta}_{\sigma_1} = \mathrm{M}\{[\eta^\alpha(t_0 + t_y) - \xi^\alpha_{\sigma_1}][\eta^\beta(t_0 + t_y) - \xi^\beta_{\sigma_1}]\};$$

$$k_{\sigma_2} = \mathrm{M}(\xi_{\sigma_2} - 1)^2; \quad k^\alpha_{\sigma_1\sigma_2} = \mathrm{M}\{\xi^\alpha_{\sigma_1}(\xi_{\sigma_2} - 1)\}.$$

Theorem 3. *The statistical characteristics of the solutions of Problems II, II^{σ_1}, and II^{σ_2} are connected by the formulas*

$$\|k_R^{\alpha\beta}\| = \|k_{\sigma_1}^{\alpha\beta}\| - \|c_R^{\alpha} k_{\sigma_1\sigma_2}^{\beta} + c_R^{\beta} k_{\sigma_1\sigma_2}^{\alpha}\| + \|c_R^{\alpha} c_R^{\beta} k_{\sigma_2}\| + \|k_P^{\alpha\beta}\|, \tag{6}$$

$$m_R^{\alpha} = \bar{\eta}^{\alpha} - c_R^{\alpha}, \qquad \alpha, \beta = 1, 2, \dots, n, \tag{7}$$

where $\|k_P^{\alpha\beta}\|$ is a symmetric nonnegative definite matrix—the matrix of artificial scattering.

Let $\mathcal{P}_{\alpha\beta}^{R}(\tau)$, $\mathcal{P}_{\alpha\beta}^{\sigma_1}(\tau)$, and $\mathcal{P}_{\beta}^{\sigma_2}(\tau)$ be the weight functions corresponding to the solutions of problems II, II^{σ_1}, and II^{σ_2}, and let $w(\tau)$ be an arbitrary function, not identically zero, satisfying the conditions

$$\int_0^T M\xi^{\alpha}(t_0 - \tau) w(\tau)\, d\tau = 0, \qquad \alpha = 1, 2, \dots, n.$$

Suppose, in addition, that the parameters $\varkappa_{\alpha\beta}$ are computed from the equations

$$\sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \varkappa_{\alpha\mu} \varkappa_{\beta\nu} a_{\mu\nu} = k_P^{\alpha\beta},$$

where $\|k_P^{\alpha\beta}\|$ satisfies equation (6), and the constants $a_{\mu\nu}$ are computed from $k_{\xi^{\alpha},\xi^{\beta}}(t_1, t_2)$ and $w(\tau)$ by formulas (2).

Theorem 4. *The weight functions for which the solutions of problems II,* II^{σ_1}, *and* II^{σ_2} *are attained are connected by the relations*

$$\mathcal{P}_{\alpha\beta}^{R}(\tau) = \mathcal{P}_{\alpha\beta}^{\sigma_1}(\tau) + c_R^{\alpha} \mathcal{P}_{\beta}^{\sigma_2}(\tau) + \varkappa_{\alpha\beta} W(\tau), \qquad \alpha, \beta = 1, 2, \dots, n. \tag{8}$$

Theorems 1-4 reduce the solution of the complicated variational problems I and II to the analysis of the substantially simpler variational problems I^{σ}, II^{σ_1}, and II^{σ_2}, considered in the literature, and to the investigation for an extremum of the functions $R(k^{\alpha\beta}) = R^0(k_P^{\alpha\beta})$ and $R(m^{\alpha}, k^{\alpha\beta}) = R^0(c^{\alpha}, k_P^{\alpha\beta})$, respectively.

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CITED LITERATURE

1. D. B. Yudin, DAN, **177**, No. 3 (1967).

Note: Figure translations are in progress. See original paper for figures.