ON A METHOD FOR INVESTIGATING THE STABILITY OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

E. N. ROZENVASSER

ON A METHOD FOR INVESTIGATING THE STABILITY OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

(Presented by Academician V. I. Smirnov, 12 XII 1966)

1. We shall consider the system of equations

$$D(p)Y = \mu M(p)B(t)Y \quad (p = d/dt), \tag{1}$$

where $\mu > 0$ is a scalar parameter; Y is a vector having n components; D is a square $n \times n$ matrix; M and B are rectangular matrices $n \times m$ and $m \times n$, respectively; the matrix $B(t)$ is periodic and satisfies the Dirichlet conditions for expansion in a Fourier series:

$$B(t) = \sum_{k=-\infty}^{\infty} B_k e^{ki\omega t}, \qquad B_k = \frac{1}{T}\int_0^T B(\tau)e^{-ki\omega\tau}\,d\tau, \qquad \omega = \frac{2\pi}{T}. \tag{2}$$

The matrices D and M are polynomials in the differentiation operator $p = d/dt$

$$D(p) = \sum_{k=0}^{r} D_k p^{r-k}, \qquad M(p) = \sum_{k=0}^{r-1} M_k p^{r-1-k}. \tag{3}$$

Here D_k, M_k are constant real matrices, and $\det D_0 \neq 0$. Let the equation

$$d(p) = \det D(p) = 0 \tag{4}$$

have roots p_ρ $(\rho = 1, 2, \dots, s)$ with multiplicities l_s, and let $\operatorname{Re} p_\rho = 0$ $(\rho = 1, 2, \dots, q)$, $\operatorname{Re} p_\rho < 0$ $(\rho = q+1, \dots, s)$. We divide the roots p_ρ $(\rho = 1, 2, \dots, q)$ into groups of numbers e_α $(\alpha = 1, 2, \dots)$, mutually comparable modulo $i\omega$. The roots of each group will be denoted by $p_{\alpha m}$ $(m = 0, 1, \dots, q_\alpha)$. In this case, by definition, $p_{\alpha m} = p_{\alpha 0} + k_m^\alpha i\omega$, where the k_m^α are integers. We shall denote by

k_α the set of numbers k_m^α of one group. The set-theoretic sum of the sets k_α will be denoted by k. Our principal problem is to investigate the stability of system (1) for small $|\mu|$. This problem has been studied by many authors ($^{1-4}$). Usually the solution was obtained by applying various methods of successive approximations, associated with the multiplicity of the roots p_ρ and the form of the corresponding elementary divisors. A very general approach to this problem is set forth in (^{3}).

In the present paper a method of investigation is proposed, based on the application of one class of integral equations and suitable, generally speaking, for arbitrary values of μ. In a number of problems the proposed method makes it possible to simplify the computations connected with finding stability conditions. At the same time, application of the method does not require a preliminary reduction of system (1) to the Cauchy normal form and does not depend on the structure of the elementary divisors of the matrix $D(p)$.

We shall say that system (1) is **stable** (**unstable**) as $\mu \to 0$ if there exists an arbitrarily small $\mu^* > 0$ such that system (1) is stable (unstable) for $0 < \mu < \mu^*$.

2. Let k_H be an arbitrary finite set of integers. If $k_H = k$, then the set k_H will be called **admissible**. To each admissible-

to each such set we associate the following linear system of equations with respect to the vectors Y_k:

$$D(\lambda + ki\omega)Y_k = \mu M(\lambda + ki\omega) \sum_{s \in k_H} B_{k-s} Y_s \quad (k \in k_H), \tag{5}$$

where λ is a scalar parameter, and the number s in each of the equations runs through the whole set k_H. We denote the matrix of the homogeneous system (5) by V_H. Consider the equation

$$\det V_H = \Delta_H(\lambda, \mu) = 0, \tag{6}$$

whose roots we denote by λ_H^r $(r = 1, 2, ...)$. We shall say that equation (6) possesses a certain property in a neighborhood of the matrices B_i if this property is preserved when the matrices B_i are replaced by $B_i + \mu U_i$, where U_i are arbitrary constant matrices, for all sufficiently small $\mu > 0$.

Theorem 1. *If, for at least one choice of an admissible set k_H, in a neighborhood of the matrices B_i we have $\operatorname{Re} \lambda_H^r < 0$, then system (1) is stable as $\mu \to 0$.*

If, for at least one choice of an admissible set k_H, in a neighborhood of the matrices B_i, for at least one root $\lambda_H^{r_0}$ such that $\lim_{\lambda \to 0} \lambda_H^{r_0} = e_{\alpha 0}$, we have $\operatorname{Re} \lambda_H^{r_0} > 0$, then system (1) is unstable as $\mu \to 0$. From equation (6) one can find stability conditions independent of μ, by applying the Hurwitz conditions.

Theorem 2. *Let c_α be one of the groups of roots comparable modulo $i\omega$. Call an arbitrary numerical set $k_{H\alpha}$ admissible with respect to the group c_α if $k_{H\alpha} \supseteq k_\alpha$.*

Consider the system of equations of the form (5) with k_H *replaced by* $k_{H\alpha} \supseteq k_\alpha$. *Denote the corresponding equation (6) by*

$$\Delta_{H\alpha}(\lambda, \mu) = 0 \quad (\alpha = 1, 2, \ldots). \tag{7}$$

Let $\lambda^r_{H\alpha}$ *be the roots of equation (7). Then, if for each* α *there exists* $k_{H\alpha}$ *such that, in a neighborhood of the matrices* B_i, *we have* $\operatorname{Re}\lambda^r_{H\alpha} < 0$, *then system (1) is stable as* $\mu \to 0$.

If, for at least one α, *there exists* $k_{H\alpha}$ *such that equation (7), in a neighborhood of the matrices* B_i, *has a root* $\lambda^{r_0}_{H\alpha}$ *such that* $\operatorname{Re}\lambda^{r_0}_{H\alpha} > 0$ *and* $\lim_{\mu\to 0}\lambda^{r_0}_{H\alpha} = p_{\alpha 0}$, *then system (1) is unstable as* $\mu \to 0$.

The equations (7) appearing in Theorem 2 have, generally speaking, degree lower than the equation (6).

Let L_α be the sum of the multiplicities of the roots entering the group c_α. Then any equation of the form (7) can be reduced to an equivalent equation having degree L_α with respect to λ. Since

$$\prod_{k\in k_{H\alpha}} d(\lambda + ki\omega) = (\lambda - p_{0\alpha})^{L_\alpha}\, \bar d_{H\alpha}, \quad \bar d_{H\alpha}(p_{0\alpha}) \neq 0,$$

instead of (7) one can obtain the equation

$$(\lambda - p_{0\alpha})^{L_\alpha} + \mu \sum_{k\geq 0} \mu^k g_{k\alpha}(\lambda) + \mu f(\lambda, \mu) = 0, \tag{8}$$

where $g_{k\alpha}(\lambda)$ are polynomials in λ of degree lower than L_α; $f(\lambda, \mu)$ is a rational function of λ, with numerator degree less than denominator degree, holomorphic in a neighborhood of the point $\lambda = p_{0\alpha}$.

Theorem 3. *The assertions of Theorem 2 remain valid if, instead of (7), for each group of roots* c_α *one considers equation (8).*

3. We briefly outline the proof of Theorem 1.

Lemma. *Let* k_H *be an admissible set. If system (1) has a solution of the form*

$$Y = e^{\lambda t}\bar Y(t), \qquad \bar Y(t) = \bar Y(t + T), \tag{9}$$

where λ is a number, then the function $\overline{Y}(t)$ satisfies the system of equations

$$\overline{Y}(t) = \mu \int_0^T \Phi_b(\lambda, t-\tau) B(\tau) \overline{Y}(\tau)\, d\tau + \overline{Y}_H, \tag{10}$$

$$D(\lambda + ki\omega)Y_k = \mu M(\lambda + ki\omega)\frac{1}{T}\int_0^T B(\tau)\overline{Y}(\tau)e^{-ki\omega\tau}\,d\tau, \qquad k \in k_H, \tag{11}$$

in which

$$\overline{Y}_H = \sum_{k\in k_H} Y_k e^{ki\omega t}, \qquad Y_k = \frac{1}{T}\int_0^T \overline{Y}(\tau)e^{-ki\omega\tau}\,d\tau,$$

$$\Phi_b(\lambda, t-\tau) = \frac{1}{T}\sum_{\substack{k=-\infty \\ k\notin k_H}}^{\infty} W(\lambda + ki\omega)e^{ki\omega(t-\tau)}, \qquad W(\lambda) = D^{-1}(\lambda)M(\lambda). \tag{12}$$

The proof of the lemma follows from (5). Solving the integral equation (10), in which $\overline{Y}_H$ is temporarily assumed known, and substituting from (12), we obtain

$$\overline{Y} = \sum_{k\in k_H}[Ee^{ki\omega t} + \mu F_k(\lambda,\mu,t)]Y_k, \tag{13}$$

where E is the identity matrix,

$$F_k(\lambda,\mu,t) = \int_0^T R(\lambda,\mu,t,\tau)e^{ki\omega\tau}\,d\tau$$

and $R(\lambda,\mu,t,\tau)$ is the resolvent of equation (10). Substituting the expression into (11), we obtain a homogeneous system of equations with respect to the vectors Y_k ($k \in k_H$). The condition for solvability of this system is the vanishing of its determinant $D_H(\lambda,\mu)$. Therefore the equation $D_H(\lambda,\mu) = 0$ determines the characteristic exponents $\lambda = \lambda(\mu)$. It is easy to verify that the elements of the determinant $D_H(\lambda,\mu)$ differ from the elements of the determinant Δ_H in (6) by quantities of order μ^2. Using this circumstance, we arrive at Theorem 1, since, when its conditions are fulfilled, all roots of $D_H(\lambda,\mu) = 0$ that are characteristic exponents of system (1) have negative real parts for sufficiently small $\mu > 0$.

The proofs of Theorems 2 and 3 are carried out from analogous considerations.

4. Let us show that Theorems 1 and 2 justify the applicability of Hill’ s method to system (1). Indeed, assuming that in (9)

$$\overline{Y} = \sum_{k=-\infty}^{\infty} Y_k e^{ki\omega t}$$

and substituting this expression into (1), we formally arrive at the infinite system of equations

$$D(\lambda + ki\omega)Y_k = \mu M(\lambda + ki\omega) \sum_{s=-\infty}^{\infty} B_{k-s} Y_s \qquad (k = 0, \pm 1, \ldots) \tag{14}$$

with matrix

$$V_\infty = |D(\lambda + ki\omega)\delta_{ks} - \mu M(\lambda + ki\omega) B_{k-s}|\,, \tag{15}$$

where δ_{ks} is the matrix Kronecker symbol. If, however, we consider approximate solutions

$$\overline{Y} \approx \sum_{k \in k_H} Y_k e^{ki\omega t},$$

then we arrive at the finite system of equations (5), whose matrix is a certain "piece" of the infinite matrix V_∞. In this case, in order to obtain correspondence between the exact solution and the approximate one, it is necessary that the equality $k_H \supseteq k$ hold, i.e., that the approximate solution include all harmonics with numbers k_m^α. In the simplest case we must set $k_H = k$. If the corresponding equation (6) does not answer the question of the signs of $\operatorname{Re}\lambda(\mu)$,

then one can include a larger number of harmonics in the approximate solution and construct a new equation of the form (6). In this case it is practically expedient to choose the additional harmonics in such a way that k_H coincides with the set of all integers $-N \leq k_H \leq N$, and then to increase the number N. It can be shown that, with such a choice of the sequence k_H, one can obtain stability conditions for all μ, and not only for small values in modulus.

As an example, suppose that among the roots p_ρ $(\rho = 1, 2, \ldots, q)$ only one root enters each group c_α. Then one may put $\bar{Y} \approx Y_0 = \text{const}$. In this case equation (6) has the form $\det |D(\lambda) - \mu M(\lambda) B_0| = 0$. This equation is the characteristic equation of the system obtained from (1) by averaging over the period. Therefore we obtain the following assertion. If all roots of equation (4) with real part equal to zero are not congruent modulo $i\omega$, then the asymptotic stability or instability of system (1) as $\mu \to 0$ coincides with the asymptotic stability or instability of the matrix B_0 in a neighborhood, for the system obtained by averaging over the period.

5. Let us establish the connection between the determinant of the infinite matrix (15) and a Fredholm equation of the second kind. Denote by $u(N)$ the set of all integers from $-N$ to N. Let $k_H = u(N)$ $(N = 0, 1, \ldots)$. Then the sequence $\det V_H$ is a sequence of central minors of matrix (15). This sequence, generally speaking, does not tend to a finite limit. However, by normalizing matrix (15), one can obtain a convergent determinant. Indeed,

consider the normalized matrix $\bar{V}_\infty = |\delta_{ks} - \mu W(\lambda + ki\omega) B_{k-s}|$. Denote by $\det \bar{V}_H$ the central minors of $\bar{V}_\infty$ corresponding to the sets $k_H = u(N)$.

Theorem 4. *The sequence of determinants* $\det \bar{V}_H$ $(k_H = u(N))$ *converges as* $N \to \infty$ *for all* $\lambda \neq p_\rho + ki\omega$ $(k = 0, \pm 1, \ldots)$*. Moreover,*

$$\lim_{N \to \infty} \det \bar{V}_H = g(\lambda, \mu),$$

where $g(\lambda, \mu)$ *is the Fredholm denominator of the integral equation, defined in the square* $0 < t, \tau < T$*, with kernel*

$$k(t, \tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W(\lambda + ki\omega) e^{ki\omega(t-\tau)} B(\tau), \qquad t \neq \tau,$$

$$k(t, t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W(\lambda + ki\omega) B(t), \qquad t = \tau.$$

A special case of system (1) is the system in Cauchy normal form $\dot{Y} = AY + \mu B(t) Y$. For this system the expression $W(\lambda)$ in (13) takes the form $W(\lambda) = (E\lambda - A)^{-1}$, and the matrix $\bar{V}_\infty$ has the form

$$\bar{V}_\infty = \left| \delta_{ks} - \mu (E(\lambda + ki\omega) - A)^{-1} B_{k-s} \right|. \tag{16}$$

From Theorem 4 and (5) there follows the following assertion.

Theorem 5. *The sequence* $\det \bar{V}_H$ $(k_H = u(N))$ *for the matrix (16) converges as* $N \to \infty$ *for all* $\lambda \neq p_\rho + ki\omega$, $(k = 0, \pm 1, \ldots)$*. Moreover,*

$$\lim_{N \to \infty} \det \bar{V}_H = g(\lambda, \mu),$$

where $g(\lambda, \mu)$ *is the Fredholm denominator of the integral equation with kernel*

$$k(t, \tau) = \begin{cases} e^{-\lambda(t-\tau)} e^{A(t-\tau)} \left(E - e^{-\lambda T} e^{AT}\right)^{-1} B(\tau), & 0 < t - \tau < T, \\ e^{-\lambda(t-\tau+T)} e^{A(t-\tau+T)} \left(E - e^{-\lambda T} e^{AT}\right)^{-1} B(\tau), & -T < t - \tau < 0, \\ \dfrac{1}{2} \operatorname{cth} \dfrac{(E\lambda - A)T}{2} B(t), & t = \tau. \end{cases}$$

The justification of the method set forth above can also be carried out using Theorem 4.

Leningrad Shipbuilding Institute

Received
25 X 1966

References

1. I. G. Malkin, *Some Problems in the Theory of Nonlinear Oscillations*, 1956.

2. N. P. Erugin, *Systems of Ordinary Linear Differential Equations*, Minsk, 1963.

3. V. I. Fomin, V. A. Yakubovich, Computation of characteristic exponents of linear systems with periodic coefficients, *Methods of Computation*, issue 3, Leningrad, 1966.

4. L. Cesari, *Asymptotic Behavior and Stability of Solutions of Ordinary Differential Equations*, Moscow, 1964.

5. E. N. Rozenvasser, Proceedings of the Second All-Union Congress on Theoretical and Applied Mechanics, 1, 1965.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.