INTERPOLATION FAMILIES OF FUNCTIONS AND EMBEDDINGS OF SETS IN EUCLIDEAN AND PROJECTIVE SPACES

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Abstract

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INTERPOLATION FAMILIES OF FUNCTIONS AND EMBEDDINGS OF SETS IN EUCLIDEAN AND PROJECTIVE SPACES

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1. Let natural numbers m and $k \leq m$ be given. An **interpolation family of functions of order** m **and degree** k, or an (m, k)-**system**, will mean a collection of real functions

$$f_0(t),\ f_1(t), \dots,\ f_m(t), \tag{1}$$

continuous on some topological space T and satisfying the following condition.

(A). For any distinct points $t_0, t_1, \dots, t_k$ of the space and any real numbers $a_0, a_1, \dots, a_k$, there exists a polynomial in the system (1)

$$p(t) = \sum_{i=0}^{m} c_i f_i(t), \tag{2}$$

such that $p(t_j) = a_j$ $(j = 0, 1, \dots, k)$.

It is easy to see that condition (A) in this definition may be replaced by either of the following:

(B). For any distinct points $t_0, t_1, \dots, t_k$ of the space T, the rank of the matrix $\|f_i(t_j)\|_{i=0}^{i=m}{}_{j=0}^{j=k}$ is equal to $k + 1$.

(C). Any $m - k + 1$ linearly independent polynomials of the form (2) have on T no more than k common zeros.

As a rule, we shall assume the space T to be a bicompact Hausdorff space (a bicompactum).

Interpolation families of functions, besides being of independent interest, also arise in the problem of best approximation of continuous functions by generalized polynomials (^{1,2}), by generalized rational functions (^{3}), in the finite moment problem (4), and in the problem of finite-dimensional linear positive operators (5).

2. A subset X of the projective space P^m is called k-regular if every $(k-1)$-dimensional (projective) plane in P^m meets this set in at most k points. We shall call any homeomorphic mapping of a topological space T onto a k-regular subset X of P^m a k-regular embedding. A k-regular subset of the Euclidean space E^m and a k-regular embedding into this space are defined analogously.

If an (m,k)-system of functions (1) is given on a bicompactum T, then, by assigning to each point $t \in T$ the point of projective space P^m with homogeneous coordinates $f_0(t), f_1(t), \dots, f_m(t)$, we obtain a k-regular embedding $F : T \to P^m$. Conversely, if there is some k-regular embedding of the space T in P^m, then the homogeneous coordinates of a variable point of the image give an (m,k)-system of functions defined on T.

Sometimes one has to consider interpolation systems (1) for which $f_0(t) \equiv 1$. Such systems define a k-regular embedding of the space T in the Euclidean space E^m.

It follows from what has been said, in particular, that a bicompactum on which an (m,k)-system of functions exists is metrizable (i.e. is a compactum) and has finite dimension, which, as can be shown, does not exceed $m-k+1$.

3. Denote by $D_k(T)$ (respectively, by $d_k(T)$) the least dimension of a projective (respectively, Euclidean) space into which the compactum T is k-regularly embeddable. Clearly, $D_k(T) \leq d_k(T)$. Apparently, the question of computing $D_k(T)$ for $k > 1$ has not previously been considered. As for the number $d_k(T)$, its exact value is known only in a few cases. In papers (6–10), upper and lower estimates for it were obtained for various classes of compacta. Here we indicate new estimates of the numbers $D_k(T)$ and $d_k(T)$ for certain spaces T. As consequences, some relations are obtained between the orders and degrees of interpolation systems of functions on these spaces.

The lower estimates for the number $D_k(T)$ are based on the following theorem.

Theorem 1. *Let a compactum T be k-regularly embedded in the projective space P^m, and let $t_1, t_2, \dots, t_i$ be distinct points of this compactum, their number being $i = [(k+1)/2]$. Then there exist closed neighborhoods $M_1, M_2, \dots, M_i$ of these points relative to T, mutually disjoint and such that the direct product*

$$M_1 \times M_2 \times \dots \times M_i \times S,$$

where S is a simplex of dimension $[k/2]$, is homeomorphically embeddable in the Euclidean space E^m.

Theorem 2. *For the n-dimensional cube I^n the inequality*

$$D_k(I^n) \geq [(k+1)/2](n-1)+k$$

holds.

Theorem 3. *If a topological space contains an n-dimensional cube, then on it there cannot exist an (m,k)-system of even degree k satisfying the inequality*

$$k > 2m/(n+1),$$

or of odd degree k, satisfying the inequality

$$k > 2m/(n+1)-(n-1)/(n+1).$$

We shall call an n-dimensional umbrella a set T_n lying in the space E^{n+1} with coordinates $(t_1, t_2, \dots, t_{n+1})$ and consisting of the n-dimensional ball

$$t_1^2+t_2^2+\dots+t_n^2 \leq 1, \qquad t_{n+1}=0$$

and the segment

$$t_1=t_2=\dots=t_n=0, \qquad 0 \leq t_{n+1} \leq 1,$$

as well as any set homeomorphic to T_n.

Theorem 4. *The direct product of p umbrellas of dimension n is not topologically embeddable in the Euclidean space E^{np+p-1}.*

For $n=1$ this assertion was proved in ([9]).

Theorem 5. *If a finite-dimensional compactum T contains p umbrellas of dimension n and if $p \leq [(k+1)/2]$, then the inequality*

$$D_k(T) \geq [(k+1)/2](n-1)+k+p.$$

holds.

Theorem 6. *Let a topological space contain p umbrellas of dimension n. Then on it there cannot exist (m,k)-systems of even degree k, satisfying the inequalities*

$$k \geq 2p, \qquad k > 2(m-p)/(n+1),$$

or of odd degree k, satisfying the inequalities

$$k \geq 2p-1, \qquad k > 2(m-p)/(n+1)-(n-1)/(n+1).$$

4. We indicate an upper estimate for the number $d_{2k}(E^n)$. The derivation of this estimate is based on the following geometric theorem of Radon (^{11}): *every set of*

$m+2$ points lying in an m-dimensional linear space can be divided into two nonempty nonintersecting subsets whose convex hulls have a common point. It follows from Radon' s theorem that the system of continuous functions

$$f_0(t)=1, \quad f_1(t),\dots,f_m(t), \tag{3}$$

defined on the space T, will be an interpolation system of degree $2k$ in the case when it has the following property:

(Γ_k). For any s $(s \leqslant k)$ points of the space T there exists a polynomial in the system (3) that is equal to zero at these points and positive on their complement.

Indeed, in the contrary case, in the space E^m there is a plane E^{2k-1} containing distinct points

$$x_1, x_2, \dots, x_{2k+1} \tag{4}$$

of the image $F(T)$ of the set T under the mapping determined by the system (3). By Radon' s theorem the set of points (4) can be represented (with a suitable choice of the numbering of these points) as the union of two nonintersecting subsets $\{x_1,\dots,x_s\} \cup \{x_{s+1},\dots,x_{2k+1}\}$, $s \leqslant k$, whose convex hulls intersect. On the other hand, by virtue of the property (Γ_k), in the space E^m there exists a hyperplane E^{m-1} supporting the set $F(T)$ and such that $E^{m-1} \cap F(T) = \{x_1,\dots,x_s\}$.

Returning to the convex hulls of the sets $\{x_1,\dots,x_s\}$ and $\{x_{s+1},\dots,x_{2k+1}\}$, we see that the first of them lies in the hyperplane E^{m-1}, while the second lies in one of the open half-spaces determined by this hyperplane. Consequently, these convex hulls cannot intersect.

Consider on the space E^n the system of functions

$$\begin{gathered} 1,\ t_i,\ t_{i_1}t_{i_2},\ t_{i_1}t_{i_2}t_{i_3},\dots,t_{i_1}t_{i_2}\cdots t_{i_k}, \\ t_{i_1}t_{i_2}\cdots t_{i_{k-1}}(t_1^2+t_2^2+\cdots+t_n^2),\ t_{i_1}t_{i_2}\cdots t_{i_{k-2}}(t_1^2+t_2^2+\cdots+t_n^2)^2,\dots, \\ \dots,\ t_{i_1}t_{i_2}(t_1^2+t_2^2+\cdots+t_n^2)^{k-2},\ t_i(t_1^2+t_2^2+\cdots+t_n^2)^{k-1}, \\ (t_1^2+t_2^2+\cdots+t_n^2)^k, \end{gathered} \tag{5}$$

where $(t_1,t_2,\dots,t_n)$ are the coordinates of the variable point of this space and each index $i, i_1, i_2, \dots, i_k$, independently of the others, takes the values $1,2,\dots,n$. This system of functions has the property (Γ_k). Indeed, if $\rho(x,y)$ is the metric in E^n and $x_1,x_2,\dots,x_k$ are arbitrary points of this space, then the product

$$\rho^2(x, x_1)\rho^2(x, x_2)\cdots\rho^2(x, x_k)$$

is precisely the polynomial in the system (5) whose existence is required by the property (Γ_k). Thus the constructed system of functions is interpolation and has degree $2k$. The order of this system, as is easy to see, is equal to the number $C_{n+k}^{k} + C_{n+k-1}^{k-1} - 1$. Hence,

$$d_{2k}(T) \leqslant C_{n+k}^{k} + C_{n+k-1}^{k-1} - 1,$$

where T is an n-dimensional Euclidean space or any of its subsets.

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CITED LITERATURE

1. A. Haar, Math. Ann., 78, 294 (1918).
2. G. Sh. Rubinstein, DAN, 102, No. 3, 451 (1955).
3. B. Borsowski, Numer. Math., 7, No. 2, 176 (1965).
4. A. L. Garkavi, Izv. AN SSSR, ser. matem., 28, No. 3, 553 (1964).
5. Yu. A. Shashkin, UMN, 20, No. 6, 175 (1965).
6. Z. Łuszczki, Bull. Acad. Pol. Sci., 6, 375 (1958).
7. V. G. Boltyanskii, Izv. AN SSSR, ser. matem., 23, 871 (1959).
8. V. G. Boltyanskii, S. S. Ryshkov, Yu. A. Shashkin, UMN, 15, No. 6, 125 (1960).
9. Yu. A. Shashkin, UMN, 18, No. 4, 195 (1963).
10. Yu. A. Shashkin, Izv. AN SSSR, ser. matem., 29, No. 5, 1085 (1965).
11. J. Radon, Math. Ann., 83, 113 (1921).

Note: Figure translations are in progress. See original paper for figures.

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