AN EXTREMAL PROBLEM AND LINEAR METHODS OF SUMMATION OF FOURIER SERIES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.522

MATHEMATICS

P. P. KOROVKIN

AN EXTREMAL PROBLEM AND LINEAR METHODS OF SUMMATION OF FOURIER SERIES

(Presented by Academician V. I. Smirnov on 1 II 1967)

Let $\{h_k\}_0^\infty$ be a sequence of real numbers. Denote by $m_s(h_0, h_1, \dots, h_s)$ the minimum of the function

$$z_s = \sum_{k=0}^{s}(x_k^2 + y_k^2)$$

subject to the constraint equations

$$\sum_{k=0}^{s}(x_k^2 - y_k^2) = h_0, \qquad 2\sum_{k=0}^{s-i}(x_{kx_{k+i}} - y_{ky_{k+i}}) = h_i, \quad i = 1, 2, \dots, s.$$

Theorem. If the series

$$\sum_{i=0}^{\infty} h_i \cos ix = f(x)$$

converges uniformly on the real axis, then

$$m(h_0, h_1, \dots) = \lim_{s\to\infty} m_s(h_0, h_1, \dots, h_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\, dx.$$

Proof. Since

$$u_s(x) = \left|\sum_{k=0}^{s} x_{ke}^{ikx}\right|^2 = \sum_{k=0}^{s} x_k^2 + 2\sum_{i=1}^{s}\sum_{k=0}^{s-i} x_{kx_{k+i}} \cos ix,$$

$$v_s(x) = \left|\sum_{k=0}^{s} y_{ke}^{ikx}\right|^2 = \sum_{k=0}^{s} y_k^2 + 2\sum_{i=1}^{s}\sum_{k=0}^{s-i} y_{ky_{k+i}} \cos ix,$$

the constraint equations may be rewritten as

$$u_s(x) - v_s(x) = \sum_{i=0}^{s} h_i \cos ix.$$

Taking into account the positivity of the trigonometric polynomials $u_s(x)$ and $v_s(x)$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{i=0}^{s} h_i \cos ix\right| dx \le \frac{1}{2\pi}\int_{-\pi}^{\pi}[u_s(x) + v_s(x)]\, dx = \sum_{k=0}^{s}(x_k^2 + y_k^2) = Z_s.$$

It follows from this inequality that

$$m_s(h_0, h_1, \dots, h_s) \ge \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{i=0}^{s} h_i \cos ix\right| dx,$$

$$\lim_{s\to\infty} m(h_0, h_1, \dots, h_s) \ge \lim_{s\to\infty}\int_{-\pi}^{\pi}\left|\sum_{i=0}^{s} h_i \cos ix\right| dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|\, dx. \quad (1)$$

To obtain the reverse relation, set

$$f_+(x) = (f(x) + |f(x)|)/2.$$

The function $f_+(x)$ is continuous and positive. Consequently, there exists a trigonometric polynomial $\tilde{u}_s(x)$ of order s such that the relations

$$1)\ \ \tilde{u}_s(x) > f_+(x); \qquad 2)\ \ \tilde{u}_s(x) - f_+(x) < \varepsilon, \ \ \varepsilon > 0.$$

hold. Put

$$\tilde{v}_n(x) = \tilde{u}_s(x) - \sum_{k=0}^{n} h_k \cos kt.$$

The sequence of functions $\tilde{v}_n(x)$ converges uniformly to the positive function $\tilde{u}_s(x) - f(x)$. Thus, for all sufficiently large indices n, $n \ge s$, the inequalities

$$0 \leq \tilde{v}_n(x) < \tilde{u}_s(x) - f(x) + \varepsilon.$$

will hold.

By Fejér' s theorem ([1]), the positive trigonometric polynomials $\tilde{u}_s(x)$ and $\tilde{v}_n(x)$ can be written in the form

$$\tilde{u}_s(x) = \left|\sum_{k=0}^{s} \tilde{x}_k e^{ikx}\right|^2 = \left|\sum_{k=0}^{n} \tilde{x}_k e^{ikx}\right|^2, \qquad \tilde{x}_k = 0, \quad k > s,$$

$$\tilde{v}_n(x) = \left|\sum_{k=0}^{n} \tilde{y}_k e^{ikx}\right|^2.$$

For these polynomials the connection equations are valid:

$$\tilde{u}_s(x) - \tilde{v}_n(x) = \sum_{k=0}^{n} h_k \cos kx,$$

i.e.,

$$\sum_{k=0}^{n} (\tilde{x}_k^2 - \tilde{y}_k^2) = h_0, \qquad 2\sum_{k=0}^{n-i} (\tilde{x}_k \tilde{x}_{k+i} - \tilde{y}_k \tilde{y}_{k+i}) = h_i, \quad i = 1, 2, \dots, n.$$

Putting $f_-(x) = f(x) - f_+(x)$, we obtain

$$\sum_{k=0}^{n} (\tilde{x}_k^2 + \tilde{y}_k^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\tilde{u}_s(x) + \tilde{v}_n(x)]\, dx \leq$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} [f_+(x) + \varepsilon + f_-(x) + 2\varepsilon]\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx + 3\varepsilon.$$

It follows that, for all sufficiently large indices n, the inequality

$$m_n(h_0, h_1, \dots, h_n) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx + 3\varepsilon$$

will be true. Since $\varepsilon > 0$ is arbitrary, we have

$$\overline{\lim_{n \to \infty}}\, m(h_0, h_1, \dots, h_n) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx, \tag{2}$$

and the theorem follows from relations (1) and (2).

Since every trigonometric polynomial can also be regarded as a uniformly convergent trigonometric series, we have

$$m(h_0, h_1, \dots, h_n, 0, 0, \dots) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n} h_k \cos kx \right| dx. \tag{3}$$

Relying on S. N. Bernstein's theorems on the order of growth of the derivatives of a trigonometric polynomial and Jackson's theorem on the order of approximation of functions by trigonometric polynomials, it is easy to show that, for

$$|h_k| \le M < \infty$$

$$|m_{n^2}(h_0, h_1, \dots, h_n, 0, 0, \dots, 0) - m(h_0, h_1, \dots, h_n, 0, 0, \dots)| \le c.$$

The quantity c does not depend on the number n.

By this equality the problem of estimating the integral (3), which is of importance in the theory of summation of Fourier series, is reduced to the algebraic problem of estimating the quantity $m_{n^2}(h_0, \dots, h_n, 0, \dots, 0)$.

Corollary. If $f(x)$ is an even function of class $\mathcal{L}_p$, $p > 1$, and

$$\sum_{k=0}^{\infty} h_k \cos kx$$

is its Fourier series, then

$$\lim_{n\to\infty} m(h_0, h_1, \dots, h_n, 0, 0, \dots) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx.$$

Moscow Automobile and Road Institute

Received
15 I 1967

CITED LITERATURE

1. G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, Part II, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.