ON SPIKES AND GLINTS OF RANDOM FIELDS

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Abstract

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ON SPIKES AND GLINTS OF RANDOM FIELDS

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Let $\zeta_s = \zeta_s(\omega)$ be a real random field ([1]), where $\omega \in \Omega$, $s = (s_1, \dots, s_n) \in R^n$; $[\Omega, \mathfrak{B}_\Omega, \mathbf{P}]$ is a probability space of elementary events; R^n is n-dimensional Euclidean space. It is assumed that the field ζ_s has, with probability 1, continuous first and second partial derivatives and is homogeneous with respect to the group of parallel translations. It is also assumed that the joint distributions of $\nabla\zeta_s' = (\partial\zeta_s/\partial s_1, \dots, \partial\zeta_s/\partial s_n)$ and $\nabla\zeta_s\nabla' = \|\partial^2\zeta_s/\partial s_i \partial s_j\|$ at points $s^i = (s_1^i, \dots, s_n^i)$, $i = 1, \dots, m$, have continuous probability densities $p_s^{(1)}(v; u^1, \dots, u^m)$, $p_s^{(2)}(v; U; Z)$, where $s' = (s^1, \dots, s^m)$, $v' = (v_1, \dots, v_m)$, $v_i = \zeta_{s^i}$, $U = (u^1, \dots, u^m)$, $u^i = (u_1^i, \dots, u_m^i)$, $u_j^i = \partial\zeta_{s^i}/\partial s_j^i$, $Z = (z^1, \dots, z^m)$, $z^i = \|z_{k,l}^i\|$, $z_{k,l}^i = \partial^2\zeta_s^i/\partial s_k^i \partial s_l^i$, $k, l = 1, \dots, n$; $i = 1, \dots, m$.

The principal aim of this work is to obtain explicit expressions for factorial moments of arbitrary order for the number of spikes, glints, and other local characteristics forming random flows ([2]), generated by random fields. The formulas found are a generalization of Rice' s formulas ([3]) for the mean number of maxima of a random process.

Let $n_s' = (-\nabla\zeta_s', 1)$ be a vector parallel to the normal to the tangent plane to ζ_s at the point s, and let e_i be the unit vector specifying the direction of the i-th coordinate in the space R^{n+1} of points $(s_1, \dots, s_n, \zeta)$. The notation $A = \|a_{ij}\| < (>) 0$ means that the matrix A is negative (positive) definite.

Definition 1. We shall say that the field ζ_s at the point s has: a) a **spike**; b) a **glint, consistent with the vector** $b' = (b_1, \dots, b_n)$; c) a **saddle point exceeding the level** u, if $\zeta_s \geq u$ and the corresponding condition is satisfied: a) the vector n_s is parallel to e_{n+1} and, in a sufficiently small neighborhood $U_s(\omega)$ of the point s, $\zeta_s \geq \zeta_{s'}$, $s' \in U_s(\omega)$; b) the vector n_s' is parallel to the vector $(-b', 1)$; c) the vector n_s is parallel to the vector e_{n+1}, and the principal curvatures of the surface ζ_s at the point s are nondegenerate and have different signs.

Spikes and glints will be called **regular** if, respectively, $\nabla\zeta_s\nabla' < 0$ in case a) and $\det \nabla\zeta_s\nabla' \neq 0$ in case b).

Here we shall restrict ourselves only to the indicated characteristics a), b), c), although the method set out below also makes it possible to investigate other local characteristics. Note that spikes correspond to local maxima exceeding the level u, while glints correspond to points of specular reflection for the case of infinitely distant light source and observer. For the physical meaning of such concepts see ([4]).

The random points s^i at which the homogeneous field ζ_s has spikes, glints, and saddles form homogeneous random flows. Let $\Delta_h = \{s : 0 \le s_i \le h_i,\ i = 1, \dots, n\}$ be a rectangular parallelepiped of volume $h_1 \cdots h_n$.

Theorem 1. For a homogeneous random flow $\eta(\Delta)$, $\Delta \in \mathfrak{B}(R^n)$, there exist limits

$$0 \le \lim_{h_i \downarrow 0} \frac{\mathbf{P}\{\eta(\Delta) > 0\}}{h_1 \cdots h_n} = \lambda \le \lim_{h_i \downarrow 0} \frac{\mathbf{M}\eta(\Delta_h)}{h_1 \cdots h_n} = \mu \le \infty; \qquad (1)$$

λ is called the parameter, and μ the intensity of the homogeneous flow η. If the homogeneous flow η is ordinary, then $\lambda = \mu$.

This assertion is a generalization of results of A. Ya. Khinchin and V. S. Korolyuk [(5)].

A broad class of random flows can be interpreted as being generated by systems $S = \{s^i\}$ of noncoincident random points $s^i \in R^n$. In this case $\eta(\Delta)$ is equal to the number of points $s^i \in \Delta$. For brevity we shall call such flows **regular**. Note that ordinary flows with finite intensity are regular. To the system S associate the system of random points S^{*m}, including in it all points with coordinates of the form $(s^{i_1}, \dots, s^{i_m})$, where $s^{i_k} \in S$, $i_l \ne i_r$, $l, r = 1, \dots, m$. The system S^{*m} corresponds to the regular random flow η^{*m}, defined on sets $\Delta \subseteq R^n \times \cdots \times R^n = R^{nm}$.

Definition 2. The m-measure of a regular random flow $\eta(\Delta)$, $\Delta \subseteq R^n$, is the measure defined in the space R^{nm} by the formula

$$\mu_m(\Gamma) = \mathbf{M}\eta^{*m}(\Gamma), \qquad \Gamma \subseteq R^{nm}. \qquad (2)$$

If $\mu_m(\Gamma)$ is absolutely continuous with respect to Lebesgue measure in R^{nm}, then its density $\mu(s_1, \dots, s_m)$ with respect to Lebesgue measure will be called the m-intensity.

Theorem 2. *The m-th factorial moment of the regular random flow $\eta(\Delta)$ on the set Δ is equal to the value of its m-measure on the set $\Delta^m = \Delta \times \cdots \times \Delta$, i.e.*

$$J_{(m)}(\Delta) = \mathbf{M}\eta(\Delta) \cdots [\eta(\Delta) - m + 1] = \int_{\Delta \times \cdots \times \Delta} \mu_m(ds^1, \dots, ds^m). \qquad (3)$$

The proof of (3) follows from the correspondence between S and S^{*m} that generate the systems, and from formula (2). For comparison we point to the special case of this construction used in the works [6,7].

Introduce the notation: $ds^i = ds^i_1 \cdots ds^i_n$; $ds = ds^1 \cdots ds^m$; $dv = dv_1 \cdots dv_m$;

$$\varphi_r(\mathbf{v},\mathbf{b}) = \mathbf{M}\left\{\prod_{i=1}^{m} |\det \nabla\zeta_{s^i}\nabla'|\; I(A_r)\,/\,\zeta_{s^i} = v_i,\ \Delta\zeta_{s^i} = b,\ i = 1,\dots,m\right\} \quad (4)$$

where $I(A_r)$ is the indicator function of the ω-sets A_r:

$$A_1 = \{\omega : \nabla\zeta_{s^i} < 0, \quad i = 1,\dots,m\}, \qquad A_2 = \{\omega : \det\nabla\zeta_{s^i}\nabla' \neq 0, \quad i = 1,\dots,m\},$$

$$A_3 = \bigcap_{i=1}^{m} [\omega : \det\nabla\zeta_{s^i}\nabla' \neq 0] \setminus [\omega : \nabla\zeta_{s^i}\nabla' > 0] \cup [\omega : \nabla\zeta_{s^i}\nabla' < 0].$$

Using (4), write the following integrals:

$$J^{(r)}_{(m)}(\Delta,u) = \int_{\substack{v_i \geq u\\ i=1,\dots,m}} \left\{\int_{s^i\in\Delta} \varphi_r(\mathbf{v},0)\, p^{(1)}_s(\mathbf{v},0,\dots,0)\, ds\right\} dv \quad (5)$$

$$(r = 1,3),$$

$$J^{(2)}_{(m)}(\Delta,\mathbf{b},u) = \int_{\substack{v_i \geq u\\ i=1,\dots,m}} \left\{\int_{s^i\in\Delta} \varphi_2(\mathbf{v},\mathbf{b})\, p^{(1)}_s(\mathbf{v},\mathbf{b},\dots,\mathbf{b})\, ds\right\} dv. \quad (6)$$

Theorem 3. *Let there exist finite integrals given by formulas (5), (6) for $m \leq k+1$. Then there exist finite k-th factorial moments $J^{(r)}_{(k)}$ for the random flows of regular bursts ($r=1$), flashes ($r=2$), and saddles ($r=3$), given by formulas (5), (6).*

In proving the theorem one uses a partition of R^{nm} by parallelepipeds

$$\Delta_h(t^i) = \{s^i_j :\ |s^i_j - t^i_j| \leq h_j/2\}, \qquad t^i = (t^i_1,\dots,t^i_n).$$

If $h_j \downarrow 0$, then the coordinates of a point $t^i + h^i_\omega$ of a regular burst, flash, or saddle in the parallelepiped $\Delta_h(t^i)$ are recovered with accuracy up to

infinitely small of higher order, as the solution of the linear equation

$$\nabla_{\zeta_t^i + h_\omega^i} = \nabla_{\zeta_t^i} + \nabla_{\zeta_t^i} \nabla' h_\omega^i.$$

For homogeneous Gaussian fields the conditions for the existence of intensities are easily formulated in terms of the covariance function of the field

$$R(s^1 - s^2) = M\zeta_s^1 \zeta_s^2, \qquad M\zeta_s = 0.$$

Theorem 4. *If the covariance function $R(s)$ of a homogeneous separable Gaussian field satisfies the condition*

$$\left| \frac{\partial^4 R(s)}{\partial s_1^{\varepsilon_1} \cdots \partial s_n^{\varepsilon_n}} - \frac{\partial^4 R(0)}{\partial s_1^{\varepsilon_1} \cdots \partial s_n^{\varepsilon_n}} \right| < C \sum_{i=1}^{m} |s_i|^\delta, \qquad \varepsilon_i = 0, 2, 4, \qquad \sum_{i=1}^{n} \varepsilon_i = 4,$$

$$C,\ \delta > 0,$$

then the spikes and glints are, with probability 1, regular, and the conditions formulated at the beginning of the article are satisfied. The flows generated by spikes, glints, and saddles have finite intensities, given by formulas (5), (6) for $m = 1$ and $\Delta = \{s_i : 0 \le s_i \le 1,\ i = 1, \dots, n\}$.

The proof of the theorem is based on the study of the modulus of continuity of the Gaussian field ζ_s, from whose properties follow the regularity and ordinariness of the flows under consideration.

Theorems 3 and 4 are a generalization to fields of the results obtained in papers (6, 7).

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Note: Figure translations are in progress. See original paper for figures.