SERIES OF DIRICHLET POLYNOMIALS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.53

MATHEMATICS

V. P. GROMOV

SERIES OF DIRICHLET POLYNOMIALS IN SEVERAL COMPLEX VARIABLES

(Presented by Academician I. M. Vinogradov on 8 VI 1966)

A. F. Leont' ev has a cycle of works ([1-4]), etc., devoted to the study of sequences of Dirichlet polynomials, in which profound results in the theory of functions were obtained. The purpose of the present work is to extend some results of these works to functions of several complex variables. For simplicity we restrict ourselves to two variables, but everything that follows is also true in the general case in the corresponding formulation.

Let $\{\lambda_n\}$ and $\{\mu_n\}$, $\lambda_n \nearrow \infty$, $\mu_n \nearrow \infty$, be sequences of positive numbers satisfying the condition

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \tau_1 < \infty, \qquad \lim_{n\to\infty} \frac{n}{\mu_n} = \tau_2 < \infty.$$

Consider the sequence

$$P_{n,m}(z,s) = \sum_{i,j=1}^{p_n,q_m} d_{ij}^{(n,m)} e^{-\lambda_i z e^{-\mu_j s}} (n,m = 1,2,...). \tag{1}$$

Theorem 1. Let the sequence (1) converge uniformly in the bicylinder $D(z_0,s_0) = D_1(z_0) \times D_2(s_0)$, where $D_1(z_0)$ is a simply connected domain of the plane (z) containing the vertical segment $l_1(z_0)$ of length $2\pi\tau_1$ with center at the point z_0; $D_2(s_0)$ is a simply connected domain of the plane (s) containing the vertical segment $l_2(s_0)$ of length $2\pi\tau_2$ with center at the point s_0.

Then the sequence (1) converges uniformly inside the octant Q: $\mathrm{Re}(z) > \alpha$, $\mathrm{Re}(s) > \beta$, containing the set $l(z_0,s_0) = l_1(z_0) \times l_2(s_0)$.

Definition. We shall call the numbers $\bar{x}$ and $\bar{y}$ conjugate abscissae of holomorphy of the function $F(z,s)$ if $F(z,s)$ is holomorphic in the octant $\mathrm{Re}(z) > \bar{x}$, $\mathrm{Re}(s) > \bar{y}$ and is not holomorphic in the octant $\mathrm{Re}(z) > R_1$, $\mathrm{Re}(s) > R_2$, where $R_1 \le \bar{x}$, $R_2 < \bar{y}$ or $R_1 < \bar{x}$, $R_2 \le \bar{y}$.

Theorem 2. Under the conditions of Theorem 1, the limiting function $P(z,s)$ of the sequence (1) is holomorphic in some complete convex tube domain $T_B = B + iR^2$, i.e., there exists a hypersurface $\mathfrak{M}$—the hypersurface of conjugate abscissae of holomorphy of the function $P(z,s)$ (the boundary of the domain T_B)—satisfying the condition: if (t_0, ξ_0) is any point of the hypersurface $\mathfrak{M}$, then $P(z,s)$ is holomorphic in the octant $\mathrm{Re}(z) > \mathrm{Re}(t_0),\ \mathrm{Re}(s) > \mathrm{Re}(\xi_0)$ and is not holomorphic in the octant $\mathrm{Re}(z) > R_1,\ \mathrm{Re}(s) > R_2$, where $R_1 < \mathrm{Re}(t_0),\ R_2 \leq \mathrm{Re}(\xi_0)$ or $R_1 \leq \mathrm{Re}(t_0),\ R_2 < \mathrm{Re}(\xi_0)$.

Theorem 3. Let (t_0, ξ_0) be any point of the hypersurface $\mathfrak{M}$ of conjugate abscissae of holomorphy of the function $P(z,s)$, l_1 any segment of the line $\mathrm{Re}(z) = \mathrm{Re}(t_0)$ in the plane (z), whose length is $2\pi\tau_1$, and l_2 any segment of the line $\mathrm{Re}(s) = \mathrm{Re}(\xi_0)$ in the plane (s) of length $2\pi\tau_2$.

Then on each set of the form $l_1 \times l_2$ there lies at least one singular point of the function $P(z,s)$.

Corollary. If $\tau_1 = \tau_2 = 0$, then every point of the hypersurface of conjugate abscissas of holomorphy of the function $P(z,s)$ is singular, i.e. the domain T_B is the domain of holomorphy of the function $P(z,s)$.

Theorem 4. Under the conditions of Theorem 1, the coefficients of the sequence (1) have limiting values

$$\lim_{m,n\to\infty} d_{i,j}^{n,m} = d_{i,j} \quad (i,j = 1,2,...). \tag{2}$$

Thus, to the limiting function $P(z,s)$ there corresponds a definite double Dirichlet series

$$P(z,s) \sim \sum_{i,j=1}^{\infty} d_{i,j} e^{-\lambda_i z} e^{-\mu_j s}. \tag{3}$$

Theorem 5. There exists a sequence of partial sums of the series (3) that converges uniformly to the function $P(z,s)$ in every closed bounded part of the domain T_B.

Introduce the entire functions

$$\varphi_1(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z^2}{\lambda_i^2}\right), \qquad \varphi_2(s) = \prod_{j=1}^{\infty}\left(1 - \frac{s^2}{\mu_j^2}\right)$$

and denote

$$\delta_1 = \overline{\lim_{i\to\infty}} \frac{1}{\lambda_i} \ln\left|\frac{1}{\varphi_1'(\lambda_i)}\right|, \qquad \delta_2 = \overline{\lim_{j\to\infty}} \frac{1}{\mu_j} \ln\left|\frac{1}{\varphi_2'(\mu_j)}\right|.$$

Under our conditions $0 \le \delta_1, \delta_2 \le \infty$.

Theorem 6. If the quantities δ_1 and δ_2 are finite, then the series (3) converges to $P(z,s)$ at least in the complete tubular convex domain $T_{B_1} = B_1 + i\mathbb{R}^2$, where the base B_1 of the domain T_{B_1} is obtained by a parallel displacement of the domain B by the amount δ_1 to the right along the axis $\mathrm{Re}(z)$ and by the amount δ_2 upward along the axis $\mathrm{Re}(s)$.

Remark. Examples can be indicated in which the domain T_{B_1} is the maximal domain of convergence of the series (3).

Corollary 1. If $\delta_1 = \delta_2 = 0$, then the series (3) converges to $P(z,s)$ throughout the entire domain T_B, i.e. the boundary of convergence of the series (3) coincides with the hypersurface $\mathfrak{M}$ of conjugate abscissas of holomorphy of the function $P(z,s)$.

Corollary 2. If $\tau_1 = \tau_2 = \delta_1 = \delta_2 = 0$, then the series (3) converges to $P(z,s)$ throughout the entire domain of holomorphy of the function $P(z,s)$; this domain is complete, convex, and tubular.

In conclusion, we note that the propositions stated above make it possible to obtain, as corollaries, a number of assertions concerning functions of several complex variables represented by multiple power series. In particular, the following assertions hold.

Theorem 7. Let R_1 and R_2 be conjugate radii of convergence of the power series

$$F(z,s) = \sum_{k,p=1}^{\infty} d_{n_k m_p} z^{n_k} s^{m_p}, \tag{4}$$

where

$$\lim_{k\to\infty} \frac{k}{n_k} = \tau_1, \qquad \lim_{p\to\infty} \frac{p}{m_p} = \tau_2.$$

Then on every part of the boundary of convergence of the series (4) of the form

$$|z| = R_1, \qquad |\arg z| \le \pi\tau_1,$$
$$|s| = R_2, \qquad |\arg s| \le \pi\tau_2$$

there lies at least one singular point of the sum $F(z,s)$ of the series.

Corollary. If $\tau_1 = \tau_2 = 0$, then every point of the boundary of convergence of the series (4) is a singular point for $F(z,s)$, i.e., the domain of holomorphy of the sum $F(z,s)$ of the series will be a complete circular domain.

Theorem 8. Let the positive integers $\{n_k\}$ and $\{m_p\}$ $(k, p = 1, 2, ...)$ satisfy the condition $\tau_1 = \tau_2 = 0$, and let the sequence

$$P_{k,p}(z, s) = \sum_{i,j=1}^{\alpha_k, \beta_p} d_{ij}^{(k,p)} z^{n_i} s^{m_j} \qquad (k, p = 1, 2, ...)$$

converge uniformly in some neighborhood of the point (z_0, s_0) to the function $P(z, s)$.

Then the limits

$$\lim_{k,p\to\infty} d_{ij}^{(k,p)} = d_{ij} \qquad (i, j = 1, 2, ...)$$

exist, and the series

$$\sum_{i,j=1}^{\infty} d_{ij} z^{n_i} s^{m_j}$$

converges uniformly in some neighborhood of the point (z_0, s_0) to the function $P(z, s)$ and, consequently, is complete circular.

Corollary. If the function $F(z, s)$ is regular in some neighborhood of the point $(0, 0)$ and is represented in this neighborhood by the series

$$P(z, s) = \sum_{i,j=1}^{\infty} d_{ij} z^{n_i} s^{m_j}, \tag{5}$$

whose exponents have zero densities $\tau_1 = \tau_2 = 0$, then the domain of holomorphy of the function $P(z, s)$ coincides with the open four-dimensional domain of convergence of the series (5), i.e., it is a complete circular domain.

We note that Theorem 8 is a theorem of the same type as the following result of A. F. Leont' ev ([4]). Let the positive integers s_k $(k = 1, 2, ...)$ satisfy the condition:

$$\lim_{k\to\infty} \frac{k}{s_k} = 0$$

and let the sequence

$$P_n(z, s) = \sum_{k=1}^{p_n} \sum_{p+q=s_k} a_{p,q}^{(k,n)} z^p s^q \qquad (n = 1, 2, ...)$$

converge uniformly in some domain $|z-z_0|<\rho,\ |s-s_0|<\rho$ to the function $P(z,s)$. Then the limits $\lim_{n\to\infty} a_{p,q}^{(k,n)} = a_{p,q}^{(k)}$ exist, and the series

$$\sum_{k=1}^{\infty} \sum_{p+q+s_k} a_{p,q}^{(k)} z^p s^q$$

converges to the function $P(z,s)$ uniformly inside its domain of holomorphy.

Moscow Power Engineering Institute

Received
16 V 1966

CITED LITERATURE

1. A. F. Leont'ev, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **39** (1951).
2. A. F. Leont' ev, *DAN*, **108**, No. 2 (1956).
3. A. F. Leont' ev, *Tr. III Vsesoyuzn. matem. s'' ezda*, **3**, 218 (1958).
4. A. F. Leont' ev, *Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskoi*, **77**, issue 5, 99 (1959).

Note: Figure translations are in progress. See original paper for figures.