ON THE ASYMPTOTICS OF THE SPECTRAL FUNCTION OF THE SCHRÖDINGER OPERATOR

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Abstract

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MATHEMATICS

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ON THE ASYMPTOTICS OF THE SPECTRAL FUNCTION OF THE SCHRÖDINGER OPERATOR

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In the space $L_2(R_N)$ consider the operator L:

$$Lu = -\Delta u + V(x)u,$$

where Δ is the Laplace operator, and $V(x) \geq 0$ is a function satisfying the Hölder condition at each point $x \in R_N$. Let $\theta(\lambda; x; y)$ be the kernel of the spectral function of the operator L. (It is constructed, for example, in papers ([1-3]).) For the function $\theta(\lambda; x; y)$ the following asymptotics as $\lambda \to \infty$ is known:

$$\theta(\lambda; x; y) = \left(\frac{\sqrt{\lambda}}{2\pi r_{xy}}\right)^{N/2} J_{N/2}(r_{xy}\sqrt{\lambda}) + O(\lambda^{(N-1)/2}). \tag{1}$$

If $x \neq y$, then the remainder in formula (1) has a higher order than the first written term. Our aim is to show that in some cases a much more accurate estimate of the function $\theta(\lambda; x; y)$ is valid. This can be done, however, only by imposing rather strong requirements on the function $V(x)$. We shall try to formulate these requirements in such a form that the proof of the theorem becomes obvious.

Let $V(x)$ be a function given on R_N. We shall say that the function $V(x)$ satisfies, at the points x, y, conditions α, if the following requirements are fulfilled:

1) for any fixed $\xi \in R_N$, $\tau \in [0, 1]$, the function of the real variable t

$$v(t) = V(2\sqrt{t}\,\xi + r(\tau)); \qquad r(\tau) = x + (y - x)\tau,$$

admits analytic continuation in t to the open domain D_η:

$$-\pi/2 - \eta < \arg t < \pi/2 + \eta,$$

where η is an arbitrarily small positive number;

2) there exists a number $\varepsilon > 0$ such that for all $t \in D_\eta$ the estimate

$$\sup_{\tau\in[0;1]} \left|\exp\left[-tV(2\sqrt{t}\,\xi + r(\tau))\right]\right| \leq a(t)\exp\left[2(1-\varepsilon)|\xi|^2\right],$$

holds, where $a(t)$ is a function finite at each point of the domain D_η, and moreover

$$a(t)/|t|^{N/2+1} \to 0, \qquad |t| \to \infty,$$

uniformly with respect to the argument t, if

$$-\pi/2 - \eta < \arg t < -\pi/2 + \eta, \qquad \pi/2 - \eta < \arg t < \pi/2 + \eta;$$

3) there exists a positive number $\delta > 0$ such that for all ξ satisfying the condition $|\xi| \leq \delta$, and all $\tau \in [0;1]$, the function $v(\beta^2)$ is analytic in β in some neighborhood of the point $\beta = 0$.

Potentials satisfying conditions α exist: for example,

$$V(x) = \left(\frac{1}{1+x^2}\right)^\nu$$

satisfies conditions α.

Theorem 1. *If the potential $V(x)$ satisfies conditions α, then for the spectral function $\theta(\lambda;x;y)$ the asymptotic*

expansion

$$\theta(\lambda;x;y) = \left(\frac{\sqrt{\lambda}}{2\pi r_{xy}}\right)^{N/2}\left[\sum_{n=0}^{M} a_n(x;y)\left(\frac{r_{xy}}{2\sqrt{\lambda}}\right)^n J_{N/2-n}\left(r_{xy}\sqrt{\lambda}\right)\right] + \rho_M(\lambda;x;y) \quad (x \neq y), \tag{2}$$

where M is any positive integer,

$$|\rho_M(\lambda;x;y)| \leq C_M^{(1)}(x;y)\left(\frac{\sqrt{\lambda}}{r_{xy}}\right)^{N/2-1-M} + C_M^{(2)}(x;y)\left(\frac{\sqrt{\lambda}}{r_{xy}}\right)^{N/2}\exp[-\delta' r_{xy}\sqrt{\lambda}],$$

where $C_M^{(1)}(x;y)$ and $C_M^{(2)}(x;y)$ are certain constants, finite for any x, y, $\delta' > 0$.

The first four coefficients $a_n(x;y)$ are given by the following formulas:

$$a_0(x;y) = 1;$$

$$a_1(x;y) = -\int_0^1 V(x+(y-x)\tau)\,d\tau;$$

$$a_2(x;y) = -0.5\left(\int_0^1 V(x+(y-x)\tau)\,d\tau\right)^2 - \int_0^1 \tau(1-\tau)\Delta V(x+(y-x)\tau)\,d\tau^*,$$

$$\begin{aligned}a_3(x;y) = & -0.5\int_0^1 \tau^2(1-\tau)^2\Delta\Delta V(x+(y-x)\tau)\,d\tau \\ & + \left(\int_0^1 V(x+(y-x)\tau)\,d\tau\right)\cdot\int_0^1 \tau(1-\tau)\Delta V(x+(y-x)\tau)\,d\tau \\ & + \int_0^1\left[\int_\tau^1 \nabla V(x+(y-x)\xi)\,d\xi\right]^2 d\tau - \left(\int_0^1\int_\tau^1 \nabla V(x+(y-x)\xi)\,d\xi\,d\tau\right)^2 \\ & - \frac{1}{6}\left(\int_0^1 V(x+(y-x)\tau)\,d\tau\right)^2.\end{aligned}$$

Let us dwell on the proof of this theorem. Consider the Cauchy problem for the equation:

$$\partial u/\partial t = -Lu; \qquad u(t;x)\in L_2(R_N); \qquad u|_{t=0} = u_0(x). \tag{3}$$

Lemma 1. The Green function of problem (3) is given by the formula

$$G(x;y;t) = \frac{\exp[-r_{xy}^2/4t]}{(4\pi t)^{N/2}}\times$$

$$\times\,\mathcal{E}\left\{\exp\left[-t\int_0^1 V(2\sqrt{t}(x(\tau)-\tau x(1))+x+(y-x)\tau)\,d\tau\right]\right\}, \tag{4}$$

where the symbol $\mathcal{E}\{F(x(\tau))\}$ denotes the integral with respect to Wiener measure of the functional $F(x(\tau))$; $\tau\in[0;1]$; $x(\tau)\in C_{[0;1]}(R_N)$; $x(0)=0$.

The proof of this lemma is simplest to obtain by showing, by direct calculation according to the scheme of work (5), that the function on the right-hand side of formula (4), for any $t>0$, satisfies equation (3); it is obvious that it has the required singularity as $t\to 0$. Next we show that the integral on the right-hand side of equality (4) converges for $t\in D_\eta$ and is the analytic continuation into the domain D_η of the Green function $G(x; y; t)$, which, as is known, can be computed by the formula:

$$G(x; y; t) = \int_0^\infty e^{-\lambda t}\, d_\lambda \theta(\lambda; x; y).$$

* In note ([1]) a misprint has been made in this formula.

The conditions on α make it possible, upon inverting the Laplace transform, to move the contour of integration into the left half-plane; the computation of the remaining integral is elementary.

We note that from the asymptotic formula (2) there follows quite simply the consequence: the function $f(x)$ may be arbitrarily smooth in a neighborhood of the given point x_0 (equal to 0), but its Fourier expansion in the eigenfunctions of the operator L may diverge at the point x_0. This fact for the eigenfunctions of the Laplace operator was first proved by V. A. Il' in.

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Note: Figure translations are in progress. See original paper for figures.