DYNAMIC MODEL OF THE ECONOMY

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 330.115

MATHEMATICS

Academician L. V. KANTOROVICH, I. G. GLOBENKO

DYNAMIC MODEL OF THE ECONOMY

1. In paper [1] a one-product model of the economy was considered and used for calculating the rate of efficiency. A refinement of the analysis may proceed by introducing a number of branches and products. In the present paper, while preserving the macroeconomic character of the model, we introduce two products and two departments, following in this the scheme of K. Marx. It is assumed that the output of the first department serves as the source of funds for both departments, while the output of the second department is spent on consumption. The conditions of production, expressed by the production function, are not assumed to be the same for the two departments. Denote by K_1, K_2 the funds of the first and second departments; by P_1, P_2 the net output of the first and second departments. We assume the production functions U_1 and U_2 to be given. Then

$$P_1 = U_1[K_1, T_1], \qquad P_2 = U_2[K_2, T_2],$$

where T_1, T_2 are the volumes of labor resources used respectively in the first and second departments. Here $T_1 + T_2 = T$ are the volumes of labor resources determined, for example, by a demographic function; the functions $U_i(x, y)$ $(i = 1, 2)$ are positively homogeneous of the first degree and twice differentiable, with

$$U_i'(x, 1) > 0, \qquad U_i''(x, 1) < 0 \quad (i = 1, 2), \qquad 0 < x < +\infty,$$
$$U_i(0, 1) = U_i(1, 0) = 0 \quad (i = 1, 2).$$

Consumption $V(t)$ is given a priori or represents a known function of the parameters of the system, while accumulation (the product of the first department P_1) is divided between the first and second departments.

The following hypothesis is made: funds in each department admit instantaneous convertibility, but are not transferred to the other department; therefore the volume of funds in a given department cannot decrease. Labor resources may be transferred.

Thus, we have the relations:

$$T_1 + T_2 = T; \tag{1}$$

$$U_2(K_2, T_2) = V(t); \tag{2}$$

$$K_1' + K_2' = U_1(K_1, T_1). \tag{3}$$

Further, the nontransferability of funds gives the requirement that $K_1(t), K_2(t)$ are nondecreasing functions.

Finally, differential optimization requires the fulfillment of the condition of equal-efficiency distribution of labor and funds

$$\frac{\partial U_1}{\partial K_1}\frac{\partial U_2}{\partial T_2} = \frac{\partial U_1}{\partial T_1}\frac{\partial U_2}{\partial K_2}, \quad \text{if } K_1' > 0,\ K_2' > 0. \tag{4}$$

The magnitude of the rate of efficiency determines the increase in output per unit of additional capital investment per unit of time and is equal to

$$n_\mathrm{e} = \partial P_1/\partial K_1.$$

The following cases of integrability occur:

1) $U_1(K_1, T_1) = K_1^\alpha T_1^{1-\alpha}, \quad U_2 = K_2^\beta T_2^{1-\beta}$ —the Cobb—Douglas case.
2) $U_1(K_1, T_1) = a_1 K_1 + b_1 T_1, \quad U_2(K_2, T_2) = a_2 K_2 + b_2 T_2.$

We note that if $T_2(t)$ is given, then $K_2(t)$ is also determined, and we find ourselves in the situation of the one-product model (1). The case in which the functions U_1, U_2 are identical also reduces to the one-product model.

2. By Euler' s theorem we have

$$\frac{\partial U_1}{\partial K_1}K_1 + \frac{\partial U_1}{\partial T_1}T_1 = U_1, \qquad \frac{\partial U_2}{\partial K_2}K_2 + \frac{\partial U_2}{\partial T_2}T_2 = U_2.$$

On the basis of relations (2), (3) we find

$$\frac{\partial U_1}{\partial K_1}K_1' + \frac{\partial U_1}{\partial T_1}T_1' = U_{1t}', \qquad \frac{\partial U_2}{\partial K_2}K_2' + \frac{\partial U_2}{\partial T_2}T_2' = V'(t), \qquad K_1' + K_2' = U_1.$$

Hence, using also (4), we obtain the formula

$$n = \frac{V(U_1T_1' - U_{1t}'T_1) + U_1(VT_2' - V'T_2)}{V(k_1T' - U_1T_1) + V'(K_2T_1 - K_1T_2)},$$

which in the Cobb–Douglas case takes the form

$$n = aU_1/K_1.$$

In the case where technical progress is taken into account, the expression for n takes the form

$$n = \frac{P_2(P_1T_1' - P_{1t}'T_1 + \rho_1P_1T_1) + P_1(P_2T_2' - P_{2t}'T_2 + \rho_2P_2T_2)}{P_2(K_1T' - P_1T_1) + (P_{2t}' - \rho_2P_2)(K_2T_1 - K_1T_2)},$$

where

$$P_1 = e^{\rho_1 t}U_1(K_1, T_1), \qquad P_2 = e^{\rho_2 t}U_2(K_2, T_2), \qquad \rho_1, \rho_2 \geq 0.$$

3. Suppose that the limits exist

$$\lim_{t\to\infty} T_1/T = a \neq 0; \tag{5}$$

$$\lim_{t\to\infty} T'/T = \lambda; \tag{6}$$

$$\lim_{t\to\infty} V(t)/T(t) = b. \tag{7}$$

If (5)–(7) are satisfied, then it can be shown that $b < +\infty$. Further, the following cases are possible:

1) The straight line $y = \lambda x$ and the curve $y = U_1(x - c_2, a)$ intersect at two points $c_0 < c_1$ (c_2 is a root of the equation $b = U_2(x, 1-a)$, $c_2 \leq x < +\infty$). The formulas hold

$$\lim_{t\to\infty} K_2/T = c_2; \tag{8}$$

$$\lim_{t\to\infty} \frac{K_1}{T} = \begin{cases} \text{a) } c_0 - c_2, \\ \text{b) } c_1 - c_2. \end{cases} \tag{9}$$

Moreover, if there exists a constant $c > c_0$ such that there are arbitrarily distant values of t for which $s(t) \geq c$, then case a) holds; otherwise case b) holds.

2) The straight line $y = \lambda x$ and the curve $y = U_1(x - c_2, a)$ have one common point. In this case formulas (8) and (9) hold, and cases a) and b) coincide.
3) The straight line $y = \lambda x$ and the curve $y = U_1(x - c_2, a)$ have no common points. In this case, beginning from some point in time, K_1 and K_2 become negative, and the model loses economic meaning.

If (5)–(7) are satisfied, then the asymptotic formulas hold

$$\lim_{t\to\infty} n = U_1'\left(\frac{c_0 - c_2}{a}, 1\right) \quad \text{if case a) holds;}$$

$$\lim_{t\to\infty} n = U_1'\left(\frac{c_1 - c_2}{a}, 1\right), \quad \text{if case b) holds.}$$

The function U_1' denotes $U_{1x}'(x, 1)$.

Received
18 VII 1967

REFERENCES CITED

1. L. V. Kantorovich, I. G. Globenko, DAN, 174, No. 3 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.