On inverse problems of the qualitative theory of differential equations

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Date: 1967-01-01T00:00:00+00:00

Abstract

A set of systems of differential equations is constructed, possessing specified singular points (foci, centers, nodes, and saddles), limit cycles, and separatrices. Bibliography: 5 items.

Full Text

Preamble

This work, published in 1967 (Vol. III, No. 10), addresses the construction of Lyapunov functions for autonomous systems of differential equations. We consider a system of the form:

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y)$$

where $P(x, y)$ and $Q(x, y)$ are defined in a domain G. We investigate the existence of a function $M(x, y)$ such that the total derivative with respect to time satisfies specific stability criteria. Following the methodologies established in [**?**, **?**, **?**], we define a set of auxiliary functions $\omega_i(x, y) = 0$ for $i = 1, \dots, s$, which characterize the geometric structure of the phase space and the boundaries of stability regions.

The construction of the Lyapunov function is based on the product of these auxiliary functions:

$$\Phi = \omega_0 \omega_1 \dots \omega_p \omega_{p+1} \dots \omega_s \omega_{s+1}$$

where $\omega_0 = 1$ and $\omega_{s+1} = 1$. For indices $i = 1, \dots, r$, the functions $\omega_i = 0$ represent algebraic curves or surfaces. For $i = p + 1, \dots, q$, we introduce linear forms $u_i = a_{i1}x + b_{i1}y + c_{i1}$ and $v_i = a_{i2}x + b_{i2}y + c_{i2}$, where the determinant $\Delta_i = a_{i1}b_{i2} - a_{i2}b_{i1} \neq 0$. These forms allow for the local linearization of the system near critical points or specific trajectories.

For the regions A_j defined by the intersection of these surfaces, we analyze the behavior of the partial derivatives of ω_i. The coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j$ are introduced to satisfy the following differential relations:

$$\frac{\partial \omega_j}{\partial x} P + \frac{\partial \omega_j}{\partial y} Q = \omega_j \left(\alpha_j \frac{\partial P}{\partial x} + \beta_j \frac{\partial P}{\partial y} + \gamma_j \frac{\partial Q}{\partial x} + \delta_j \frac{\partial Q}{\partial y} \right)$$

In the case where $\alpha_j = \delta_j = 0$ and $\beta_j = -\gamma_j$, the system exhibits specific rotational or Hamiltonian symmetries. By substituting these into the governing equations, we derive conditions for the existence of a positive definite function $F(x, y)$ that serves as a robust Lyapunov candidate.

The stability analysis further requires examining the quadratic forms associated with these derivatives. Let u_i and v_i be transformed coordinates such that:

$$S_i = a_{i1}^2 + (a_{i1} b_{i2} + a_{i2} b_{i1})(\beta_i + \gamma_i) + 2b_{i1}^2 = 0$$

If the discriminant of this form is negative, the equilibrium point is isolated and potentially stable. We define the function $g_i(x, y)$ to account for higher-order perturbations:

$$g_i(x, y) = [M(x, y) + Q_i] \omega_0 \ldots \omega_{i-1} \omega_{i+1} \ldots \omega_s$$

where $M(x, y)$ is a scaling factor. In the neighborhood $S(\Gamma_i, \epsilon)$ of a trajectory Γ_i, we require that the function $f_i(x, y)$ satisfies $|f_i(x, y)| > \eta_i$ for some $\eta_i > 0$.

For the system to be stable in the sense of Lyapunov, the derivative of the constructed function must satisfy $\frac{d\Phi}{dt} < 0$ within the domain of interest. By analyzing the interaction between the components ω_j and ω_m in the sub-regions A_{jm}, we establish that:

$$\Phi_{jm}[(\alpha_j a_j + \beta_j b_j)\omega_m + (\alpha_m a_m + \beta_m b_m)\omega_j] = M_{jm}(b_j d_m + b_m d_j)$$

This leads to a system of algebraic inequalities for the coefficients χ_{jm}, ensuring that the energy-like function decreases along the trajectories of the system. These results generalize previous findings in [**?**, **?**] regarding the construction of integral invariants and limit cycles for second-order autonomous systems.

References

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Note: Figure translations are in progress. See original paper for figures.