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Abstract

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MATHEMATICS

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ON STRONG CONTINUOUS METHODS OF SUMMATION OF TRIGONOMETRIC SERIES

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Numerous works are known on concrete strong methods of summation of Fourier series with respect to various orthonormal systems. Among recent works one may mention the investigations of Alexits, Kralik, Leindler ($^{1-4,7}$) on the strong methods of summation of Hardy, Vallée-Poussin, and others. General, so-called strong continuous methods of summation of numerical sequences were considered by Włodarski (^{5}); he investigated the field of a strong continuous method, i.e., the set of numerical sequences summable by this method.

In the present paper strong continuous methods (abbreviated s.c. methods) are considered as applied to the summation of general trigonometric series. The field of an s.c. method is studied and, in the case where the trigonometric series is a Fourier series, estimates are given for the approximation of certain classes of functions by means of s.c. averages.

Definition. A trigonometric series

$$T(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

with arbitrary numerical coefficients is **summed** by the s.c. method $A_c^{(m)}$ $(m > 0)$ to the function $f(x) \in C_{2\pi}$, if the following conditions are satisfied:

$$\sup_{-\pi\le x\le\pi} \sum_{k=0}^{\infty} a_k(z)\,|s_k(x) - f(x)|^m = B(z) \le B_f < \infty,$$

$$\lim_{z\to Z-} \sum_{k=0}^{\infty} a_k(z)\,|s_k(x) - f(x)|^m = 0$$

uniformly in $x \in [-\pi, \pi]$, where $\{a_k(z)\}_{k=0}^{\infty}$ are functions defined and continuous on $[z_0, Z]$, $Z \le +\infty$, and $s_k(x)$ is the partial sum of series (1).

From an s.c. method $A_c^{(m)}$, by means of a corresponding choice of the functions $\{a_k(z)\}_{k=0}^{\infty}$, concrete strong methods of summation may be obtained. Assuming everywhere that $a_k(z) \geq 0$, $k = 0, 1, 2, \ldots$, we show that the following assertions hold:

1. If

$$A(z) = \sum_{k=0}^{\infty} a_k(z) \leq A < \infty, \tag{2}$$

then from summability by the s.c. method $A_c^{(m)}$ there follows summability by the s.c. method $A_c^{(r)}$, where $0 < r \leq m$.

2. If $A(z) \to \alpha \neq 0$ as $z \to Z-$, then series (1) cannot have two different $A_c^{(m)}$-limits.
3. Conditions (2) and

$$\lim_{z \to Z-} \alpha_k(z) = 0, \qquad k = 0, 1, 2, \ldots, \tag{3}$$

are sufficient for regularity in the sense of Toeplitz of the s.n. method $A_c^{(m)}$, i.e., from the fact that

$$\|s_k(x) - f(x)\|_c \to 0 \quad \text{as} \quad k \to \infty$$

it follows that $f(x)$ is the $A_c^{(m)}$-limit of the series (1).

By $A_c^{|m|*}$ we shall denote the field of the s.n. method $A_c^{(m)}$, i.e., the set of trigonometric series (1) summable by this method. If in this set $A_c^{|m|*}$ one introduces the operations of addition of elements and multiplication by a scalar, assuming in doing so that the corresponding coefficients of the trigonometric series are added and multiplied by a scalar, then we have proved that the field $A_c^{|m|*}$ is a linear space.

If in $A_c^{|m|*}$ one introduces the norm

$$\|T\|_{A_c} = \sup_{z_0 \leq z < Z} \sup_{x} \left\{ \sum_{k=0}^{\infty} \alpha_k(z) |s_k(x)|^m \right\}^{1/m}, \tag{4}$$

assuming that

$$\alpha_k(z) \neq 0, \qquad k = 0, 1, 2, \ldots, \tag{5}$$

then the following holds.

Theorem 1. *Let the sequence* $\{\alpha_k(z)\}_{k=0}^{\infty}$ *satisfy conditions (2), (3), (5), and let* $\lim_{z\to Z-} A(z) = a \neq 0$. *Then the field* $A_c^{|m|*}$ $(m \geq 1)$ *with norm (4) is a Banach space.*

We give estimates for the approximation of certain classes of functions by $A_c^{(m)}$-means of trigonometric Fourier series, i.e., estimates of the means

$$\sigma_m(x,z,T,f) = \left\{ \sum_{k=0}^{\infty} \alpha_k(z) |s_k(f,x) - f(x)|^m \right\}^{1/m}.$$

Theorem 2. *Let* $f(x) \in \operatorname{Lip}\beta$ $(0 < \beta \leq 1)$, $m \geq 2$, *and let conditions (2), (3) be satisfied. Then*

$$\sigma_m(x,z,T,f) = \begin{cases} O\left([\delta(z)]^{1/m}\right), & \text{if } \beta m > 1, \\ O\left([\delta(z)]^{\beta}\right), & \text{if } \beta m < 1, \\ O\left([\delta(z)]^{\beta} |\ln \delta(z)|^{1-\beta}\right), & \text{if } \beta m = 1, \end{cases}$$

where

$$\delta(z) = \sum_{k=0}^{\infty} |\Delta \alpha_k(z)| = \sum_{k=0}^{\infty} |\alpha_k(z) - \alpha_{k+1}(z)|.$$

Turan (^{8}) showed that there exist continuous functions for which

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_k(f,x) - f(x)|^m \neq o(1) \quad \text{as } n \to \infty,$$

if $m = m(n) \uparrow \infty$ arbitrarily slowly.

It follows from Theorem 2 that, if $m = m(z) = o(\ln 1/\delta(z))$ as $z \to Z-$ and $f(x) \in \operatorname{Lip}\beta$, $0 < \beta \leq 1$, then $\sigma_m(x,z,T,f) = o(1)$ as $z \to Z-$.

Putting, in particular,

$$\alpha_k(z_n) = \begin{cases} 0, & \text{if } k > n, \\ \dfrac{1}{n+1}, & \text{if } k \leq n, \end{cases}$$

for some sequence $\{z_n\} \in [z_0, Z)$ converging to $Z-$, we obtain from the $A_c^{(m)}$-method the Hardy method, for which $\delta(z_n) = 1/(n+1)$.

Thus, if $f(x) \in \operatorname{Lip}\beta,\ 0 < \beta \leqslant 1,\ m = m(n) = o(\ln n)$, then

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(f,x)-f(x)|^{m(n)}\right\}^{m(n)} = o(1) \qquad \text{as } n\to\infty.$$

As in [6], we shall say that the function $g(x)\in W^r_\gamma H^\beta_k$ if $g(x)$ is represented in the form

$$g(x)=\sum_{i=1}^{\infty}\frac{1}{\pi i^r}\int_{-\pi}^{\pi}\varphi(x+t)\cos\left(it+\frac{\gamma\pi}{2}\right)dt,$$

where $r\geqslant 0$,

$$\int_{-\pi}^{\pi}\varphi(x)\,dx=0,$$

$\varphi(x)\in H^\beta_k$, i.e., its modulus of smoothness of order $k\geqslant 1$ satisfies the conditions

$$\omega_k(h,\varphi)\leqslant h^\beta, \qquad h\geqslant 0, \qquad 0<\beta\leqslant 1.$$

Theorem 3. Let $f(x)\in W^r_\gamma H^\beta_1$ $(0<\beta\leqslant 1,\ r>0)$, $m\geqslant 1$, and let conditions (2), (3) be fulfilled. Then:

1. If $0<\beta<1$, $\lim_{k\to\infty}\alpha_k(z)(k+1)^{1-m(r+\beta)}=0$, then

$$\sigma_m(x,z,T,f)=O\left\{\left(\sum_{k=0}^{\infty}\left|\Delta\frac{\alpha_k(z)}{(k+1)^{rm}}\right|((k+1)^{1-\beta m})^{1/m}\right)+\left(\sum_{k=0}^{\infty}\frac{\alpha_k(z)}{(k+1)^{(r+\beta)m}}\right)^{1/m}\right\}.$$

2. If $\beta=1$, then

$$\sigma_m(x,z,T,f)=O\left\{\left(\sum_{k=1}^{\infty}\left|\Delta\frac{\alpha_k(z)}{(k+1)^{rm}}\right|((k+1)^{1-m}\ln^m(k+1))^{1/m}+\left(\sum_{k=0}^{\infty}\frac{\alpha_k(z)}{(k+1)^{(r+1)m}}\right)^{1/m}\right\}.$$

We shall give one result concerning summability almost everywhere by strong methods. In this connection, the series (1) is summable almost everywhere by the strong method $A^{(m)}_p$ to $f(x)\in L_p$, $1\leqslant p<\infty$, if the relations

$$\sup_{z_0\leqslant z<Z}\left\{\sum_{k=0}^{\infty}\alpha_k(z)\,|s_k(x)-f(x)|\right\}^{1/m}\in L_p(-\pi,\pi),$$

$$\lim_{z\to Z-}\sum_{k=0}^{\infty}\alpha_k(z)\left|s_k(x)-f(x)\right|^m=0$$

hold for almost all x.

Theorem 4. If the conditions of Theorem 1 are fulfilled, then the field A_p^m is a Banach space with norm

$$\|T\|_{A_p}=\left\{\int_0^{2\pi}\left(\sup_z\left[\sum_{k=0}^{\infty}\alpha_k(z)\left|s_k(x)\right|^m\right]^{1/m}\right)^p dx\right\}^{1/p}.$$

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