VARIATIONAL PRINCIPLES OF THE DYNAMICS OF SHELL THEORY

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VARIATIONAL PRINCIPLES OF THE DYNAMICS OF SHELL THEORY

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In formulating dynamical problems of the theory of elastic shells in variational form, Hamilton' s principle is used, which assumes that the positions of the shell at the end of the motion are known (see, for example, [(1–6)]). However, dynamical problems of shell theory are almost always presented as problems with initial conditions. In the present note a variational principle is given for such problems with initial conditions, in particular for the case in which the motion of the shell is described by linear equations of a theory of the Timoshenko type.

Let $\mathbf{r}$ be the radius vector of a point of the middle surface of the shell; $\mathbf{n}$ the unit vector of the normal to the middle surface; x^α $(\alpha = 1, 2)$ Gaussian coordinates; $\mathbf{r}_\alpha$ the base vectors; $a_{\alpha\beta}$, $b_{\alpha\beta}$ the tensors of the first and second quadratic forms of the middle surface; z the distance from the middle surface. The radius vector of an arbitrary point of the shell is

$$\mathbf{R} = \mathbf{r}(x^1, x^2) + z\mathbf{n}(x^1, x^2). \qquad (1)$$

Next let the displacement vector of points of the shell $\mathbf{U}$ be represented in the form

$$\mathbf{U} = (v_\alpha + z\varphi_\alpha)\mathbf{r}^\alpha + w\mathbf{n}, \qquad (2)$$

where $v_\alpha(x^\gamma, t)$, $w(x^\gamma, t)$ are the components of the displacement vector of the middle surface of the shell; $\varphi_\alpha(x^\gamma, t)$ are the angles of rotation of the normal.

The components of the strain tensors $\varepsilon_{\alpha\beta}$, $\chi_{\alpha\beta}$ and of the kinematic vector ω_α can be expressed in the form

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha - 2b_{\alpha\beta} w),$$

$$\chi_{\alpha\beta} = \tfrac{1}{2}(\nabla_\alpha \varphi_\beta + \nabla_\beta \varphi_\alpha), \qquad \omega_\alpha = \varphi_\alpha + \nabla_\alpha w + b_{\alpha\beta} v^\beta, \qquad (3)$$

where ∇_α denotes covariant differentiation.

Denote the velocities and momenta respectively by

$$\vartheta_\alpha = \dot{v}_\alpha, \qquad \lambda = \dot{w}, \qquad \psi_\alpha = \dot{\varphi}_\alpha; \tag{4}$$

$$\theta_\alpha = \rho h \vartheta_\alpha, \qquad \Lambda = \rho h \lambda, \qquad \Psi_\alpha = \tfrac{1}{12}\rho h^3 \psi_\alpha. \tag{5}$$

Here ρ is the density of the shell material, and h is the shell thickness.

Let $T^{\alpha\beta}$, $M^{\alpha\beta}$, N^α be the components of the tensors of tangential forces, moments, and the vector of transverse forces; p_α, p, m_α the components of the prescribed vectors of external forces and moments.

The equations of motion will be

$$\begin{gathered}\nabla_\alpha T^{\alpha\beta} - b_\alpha^{\ \beta} N^\alpha - \dot{\theta}^\beta + p^\beta = 0,\\ \nabla_\alpha N^\alpha + b_{\alpha\beta} T^{\alpha\beta} - \dot{\Lambda} + p = 0,\\ \nabla_\alpha M^{\alpha\beta} - N^\beta - \dot{\Psi}^\beta + m^\beta = 0.\end{gathered} \tag{6}$$

We shall represent the elasticity relations in the form

$$T^{\alpha\beta} = BE^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}, \qquad N^\alpha = G\omega^\alpha, \qquad M^{\alpha\beta} = DE^{\alpha\beta\gamma\delta}\chi_{\gamma\delta}. \tag{7}$$

where

$$\begin{gathered}B = Eh/(1-\nu^2), \qquad G = Eh/8k(1+\nu), \qquad D = Eh^3/12(1-\nu^2),\\ E^{\alpha\beta\gamma\delta} = a^{\alpha\gamma}a^{\gamma\delta} + \nu c^{\alpha\gamma}c^{\beta\delta},\end{gathered} \tag{8}$$

k is the shear coefficient, and $c^{\alpha\beta}$ are the components of the discriminant tensor.

Suppose that on the part C_1 of the contour of the middle surface S the static boundary conditions are prescribed:

$$T^{\alpha\beta}n_\alpha = P^\beta, \qquad N^\alpha n_\alpha = P, \qquad M^{\alpha\beta}n_\alpha = K^\beta \tag{9}$$

and on the remaining part of the contour C_2 the geometric boundary conditions are prescribed:

$$v_\alpha = v_\alpha^*, \qquad w = w^*, \qquad \varphi_\alpha = \varphi_\alpha^*. \tag{10}$$

Here P^β, P, K^β are the components of the prescribed vectors of external forces and moments on the contour; $v^*_\alpha, w^*, \varphi^*_\alpha$ are the components of the prescribed displacement vectors and rotation angles on the contour; and n_α are the components of the unit normal vector to the shell contour.

We prescribe the initial conditions in the form

$$v_\alpha\big|_{t=0} = v^0_\alpha, \qquad w\big|_{t=0} = w^0, \qquad \varphi_\alpha\big|_{t=0} = \varphi^0_\alpha; \tag{11}$$

$$\theta_\alpha\big|_{t=0} = \theta^0_\alpha, \qquad \Lambda\big|_{t=0} = \Lambda^0, \qquad \Psi_\alpha\big|_{t=0} = \Psi^0_\alpha. \tag{12}$$

Here $v^0_\alpha, w^0, \varphi^0_\alpha, \theta^0_\alpha, \Lambda^0, \Psi^0_\alpha$ are the displacements, rotation angles, and momenta at the beginning of the motion. The equations of motion of the formulated problem (3)–(7) must be integrated under the boundary and initial conditions (9)–(12).

The indicated problem can be formulated in variational form. Denote

$$g * f = \int_0^\tau g(x^\gamma, t) f(x^\gamma, \tau - t)\, dt \tag{13}$$

and introduce into consideration the functional

$$\begin{aligned} I = \int_S \Big\{ &-\frac{1}{2}E^{\alpha\cdot\gamma\delta}(B\varepsilon_{\alpha\beta} * \varepsilon_{\gamma\delta} + D\chi_{\alpha\beta} * \chi_{\gamma\delta}) - \frac{1}{2}G\omega^\alpha * \omega_\alpha \\ &+ T^{\alpha\beta} * \Big[\varepsilon_{\alpha\beta} - \frac{1}{2}(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha - 2b_{\alpha\beta}w)\Big] + M^{\alpha\beta} * \Big[\chi_{\alpha\beta} - \frac{1}{2}(\nabla_\alpha\varphi_\beta + \nabla_\beta\varphi_\alpha)\Big] \\ &+ N^\alpha * (\omega_\alpha - \varphi_\alpha - \nabla_\alpha w - b^\beta_\alpha v_\beta) - \frac{1}{2}\rho h(\theta^\alpha * \theta_\alpha + \lambda * \lambda + \frac{1}{12}h^2\psi^\alpha * \psi_\alpha) \\ &+ \theta^\alpha * (\vartheta_\alpha - \dot{v}_\alpha) + \Lambda * (\lambda - \dot{w}) + \Psi^\alpha * (\psi_\alpha - \dot{\varphi}_\alpha) \\ &+ p^\alpha * v_\alpha + p * w + m^\alpha * \varphi_\alpha + [\theta^\alpha_0 v_\alpha + \Lambda_0 w + \Psi^\alpha_0\varphi_\alpha]_{t=\tau} \\ &- [\theta^\alpha|_{t=\tau}(v_\alpha - v^0_\alpha)|_{t=0} + \Lambda|_{t=\tau}(w - w^0)|_{t=0} + \Psi^\alpha|_{t=\tau}(\varphi_\alpha - \varphi^0_\alpha)|_{t=0}]\Big\}\, dS \\ &+ \int_{C_1} (P^\alpha * v_\alpha + P * w + K^\alpha * \varphi_\alpha)\, dC \\ &+ \int_{C_2} [T^{\alpha\beta} * (v_\alpha - v^*_\alpha) + N^\beta * (w - w^*) + M^{\alpha\beta} * (\varphi_\alpha - \varphi^*_\alpha)]\, n_\beta\, dC. \end{aligned} \tag{14}$$

By the true motion of the shell we mean the motion that occurs according to equations (3)–(7) and conditions (9)–(12). In this case the following variational principle holds.

The true motion of the shell in the time interval $(0, \tau)$ is such that the functional I has a stationary value, i.e.

$$\delta I = 0. \tag{15}$$

Varying the functions $v_\alpha, w, \varphi_\alpha, \varepsilon_{\alpha\beta}, \omega_\alpha, \chi_{\alpha\beta}, \vartheta_\alpha, \lambda, \psi_\alpha, T^{\alpha\beta}, N^\alpha, M^{\alpha\beta}, \theta_\alpha, \lambda, \Psi_\alpha$, it is not difficult to verify that the Euler–Lagrange equations and the natural boundary conditions of the functional (14) are equations (3)–(7) and conditions (9)–(12).

From the general variational principle presented above one can obtain all possible variational principles of the dynamics of shell theory, if part of relations (3) –(7), (9)–(12) are satisfied beforehand and, with their aid, the corresponding varied quantities are eliminated from functional (14). In particular, if one assumes that all functions except $v_\alpha, w, \varphi_\alpha$ have been eliminated through relations (3)–(5), (7), and that the functions $v_\alpha, w, \varphi_\alpha$ satisfy the boundary and initial conditions (10), (11), then the functional I takes the form

$$\begin{aligned} I = \int_S \{ & - \tfrac{1}{2} E^{\alpha\beta\gamma\delta} (B e_{\alpha\beta} * e_{\gamma\delta} + D \chi_{\alpha\beta} * \chi_{\gamma\delta}) - \tfrac{1}{2} G \omega^\alpha * \omega_\alpha \\ & - \tfrac{1}{2} \rho h (\vartheta^\alpha * \vartheta_\alpha + \lambda * \lambda + \tfrac{1}{2} h^2 \psi^\alpha * \psi_\alpha) + p^\alpha * v_\alpha + p * w + m^\alpha * \varphi_\alpha \\ & + [\theta_0^\alpha v_\alpha + \Lambda_0 w + \Psi_0^\alpha \varphi_\alpha]_{t=t_0} \} \, dS + \int_{C_1} (P_*^\alpha v_\alpha + P_* w + K_*^\alpha \varphi_\alpha) \, dC. \end{aligned} \tag{16}$$

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