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DIVISION ALGEBRAS OVER THE FIELD OF REAL NUMBERS

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The problem of describing division algebras of finite rank over the field of real numbers goes back to Frobenius, who gave a description of such algebras under the additional assumption of associativity. A generalized Frobenius theorem is also known, covering the case of alternative division algebras ([1]). In the present paper certain classes of division algebras with associative powers are considered. With the aid of topological considerations it is possible to prove the existence, in the algebras under consideration, of a certain special basis; under some additional assumptions this makes it possible to solve the isomorphism problem for these algebras.

1. Definition. An algebra A over a field F is called a **division algebra** if A is an algebra with identity and the equations

$$ax = b, \qquad ya = b$$

are uniquely solvable in A for arbitrary $a \neq 0,\ b \in A$.

Below, the field of real numbers R will be considered as F. The set of all division algebras having finite dimension (rank) n over R will be denoted by D_n. Classical examples of division algebras are the field R itself, the field of complex numbers, the algebra of quaternions, and the Cayley-Dickson algebra, whose dimensions over R are respectively $1, 2, 4, 8$. It is known ([2–4]) that these dimensions are the only ones for which the set D_n is nonempty. Thus the problem arises of describing the sets D_n for $n = 1, 2, 4, 8$. The problem is trivial for $n = 1, 2$, since the only division algebras of the indicated dimensions over R are, respectively, the field of real numbers and the field of complex numbers. The sets D_4, D_8 are infinite and require parametrization; the parameters may be the structure constants of the algebras under consideration. It is convenient to regard D_n as a subset of the set $\mathfrak{A}_n$ of all n-dimensional algebras with identity over R. Let $A \in \mathfrak{A}_n$; $e_0 = 1, e_1, e_2, \dots, e_{n-1}$ be some basis of A. Then multiplication in A is determined by formulas of the form

$$e_i e_j = \sum_{k=0}^{n-1} c_{ij}^k e_k, \qquad i, j = 1, 2, \dots, n-1. \tag{1}$$

The structure constants c_{ij}^k will be regarded as the coordinates of a certain point in a p-dimensional Euclidean space R_p, $p = n(n-1)^2$; then to each point of R_p there corresponds a certain algebra from $\mathfrak{A}_n$. Introduce in R_p an equivalence relation θ, considering two points of R_p equivalent if the corresponding algebras from $\mathfrak{A}_n$ are isomorphic. The set $\mathfrak{A}_n$ is in one-to-one correspondence with the quotient space R_p/θ. In R_p/θ (and hence also in $\mathfrak{A}_n$) the structure of a topological space is defined in the natural way: a subset of R_p/θ is open if and only if it is the image of an open subset of R_p under the canonical mapping $R_p \to R_p/\theta$. The space $\mathfrak{A}_n$ repre-

is a union of a finite number of subsets homeomorphic to Euclidean spaces of various dimensions; the greatest of these dimensions will be called the dimension of $\mathfrak{A}_n$. (Heuristic considerations show that the dimension of $\mathfrak{A}_n$ is equal to $n(n-1)(n-2)$.)

Theorem 1. *The set D_n is open in $\mathfrak{A}_n$, and the dimension of D_n coincides with the dimension of $\mathfrak{A}_n$* $(n = 1, 2, 4, 8)$.

For the proof of the theorem one uses deformations ([5]) of the algebra $A \in D_n$. Under arbitrary sufficiently small changes of the structural constants c_{ij}^k of the algebra A with multiplication table (1), the property of the algebra A of being a division algebra is preserved.

Corollary. *If any algebra from D_n satisfies a certain identical relation, then every algebra from $\mathfrak{A}_n$ satisfies the same identical relation* $(n = 1, 2, 4, 8)$.

2. An important class of algebras is formed by algebras with associative powers. Denote by $\mathfrak{B}_n$ the set of algebras from D_n satisfying the additional condition of associativity of powers. Each algebra from $\mathfrak{B}_n$ is quadratic, i.e., for any element x of an algebra $A \in \mathfrak{B}_n$, the degree of its minimal polynomial does not exceed 2. Let us denote by C_n the set of all quadratic algebras from $\mathfrak{A}_n$. Let $A \in C_n$. As Dickson showed ([6]), the linear space A decomposes into a direct sum of subspaces $R + V$, where R is a one-dimensional subspace containing the identity of the algebra A, and the elements of V are characterized by the property that $x^2 \in R$, $x \notin R$ for every $x \in V$. Let $x, y \in V$; then

$$xy = -(x, y) + x \cdot y, \tag{2}$$

where $(x, y) \in R,\ x \cdot y \in V$. The composition $x \cdot y$ is anticommutative and defines a linear mapping $V \otimes V \to V$; thus V has the structure of an anticommutative algebra $A^{(-)} = \langle V, +, \cdot \rangle$. Conversely, if on the linear space V a bilinear form (x, y) and an anticommutative operation $x \cdot y$ are defined, then formula (2) makes it possible to define on the direct sum of spaces $R + V$ the structure of a quadratic algebra. As in § 1, the set C_n (and $\mathfrak{B}_n$) is provided with a parametrization and becomes a topological space.

Theorem 2 ([7]). *In order that an algebra $A \in C_n$ be a division algebra, it is necessary and sufficient that the quadratic form (x, x) be positive definite and that the algebra $A^{(-)}$ satisfy condition (Q): for any linearly independent $x, y \in A^{(-)}$, the elements $x, y, x \cdot y$ also form a linearly independent system.*

An anticommutative algebra satisfying condition (Q) will be called a Q-algebra.

Theorem 3. *The set $\mathfrak{B}_n$ is open in C_n. The dimension of $\mathfrak{B}_n$ coincides with the dimension of C_n $(n = 1, 2, 4, 8)$ and is equal to $r_n = \frac{1}{2}(n-1)^2(n-2)$.*

In the case $n = 4$ the theorem follows easily from the results of ([7]). For $n = 8$ the proof of the theorem is based on the above description of algebras from $\mathfrak{B}_n$ and uses a certain class of deformations of the Cayley–Dickson algebra. From Theorem 3 it follows, in particular, that the set of algebras from $\mathfrak{B}_8$ depends on 147 "essential" parameters.

3. It is said that an algebra $A \in \mathfrak{A}_n$ is an algebra with k generators if k is the least number for which there exist such elements $a_1, a_2, \ldots, a_k \in A$ that the set $\{1, a_1, a_2, \ldots, a_k\}$ generates the algebra A. From the theorem on the dimension of division algebras over the field of real numbers it follows that every algebra from D_8 has at most three generators. If an algebra $A \in D_8$ has exactly three generators, then such an algebra is necessarily quadratic, and any linearly independent system of elements of the form $\{1, x, y\}$ generates in A a subalgebra of dimension 4. The corresponding algebra $A^{(-)}$ has an analogous property: it is generated by some triple of elements, while any two linearly independent elements generate in $A^{(-)}$ a three-dimensional subalgebra. Below

by V we shall denote an arbitrary seven-dimensional Q-algebra possessing this property.

Let H be an arbitrary two-dimensional subspace of V, and let $\{x, y\}$ be any basis of H. Then the elements x, y generate a three-dimensional subalgebra $\mathcal{H} \subset V$; this subalgebra, obviously, does not depend on the choice of basis in H. As shown in (7), any three-dimensional Q-algebra is characterized, up to isomorphism, by a single parameter β, taking values in the semi-interval $[0, 1)$; with a suitable choice of basis $\{u, v, w\}$, the multiplication table of such an algebra has the form

$$uv = w, \qquad vw = u + 2\beta v, \qquad wu = v.$$

Thus, to the subalgebra $\mathscr{H}$ there corresponds a certain number β, which we shall denote by $\beta(H)$. On the set M of all two-dimensional subspaces of the space V there is the structure of a real Grassmann manifold (of dimension 10). The function $\beta(H)$ is continuous on M and, by compactness of M, attains at some point H_0 a maximal value, which we shall again denote by β. Denote by $\mathscr{H}_0$ the subalgebra in V generated by the subspace H_0. Extending a basis of $\mathscr{H}_0$ to a basis of V, we arrive at the following theorem.

Theorem 4. In the Q-algebra V one can choose a basis u_i, v_i, w $(i = 1, 2, 3)$ such that the multiplication table of V in this basis has the form

$$u_i v_i = w, \qquad v_i w = u_i + 2\beta v_i, \qquad w u_i = v_i \quad (i = 1, 2, 3); \tag{3}$$

$$u_i u_j = -v_k, \qquad u_j v_k = -u_i, \qquad v_k u_i = -u_j, \qquad v_i v_j = v_k + 2\beta u_k,$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$; β is a uniquely determined parameter taking values in the semi-interval $[0, 1)$.

The computation of the function $\beta(H)$ is based on lemmas that make it possible to give this function an explicit algebraic expression in terms of the structure constants of the algebra V and the coordinates of the vectors x, y spanning the subspace H. In what follows the Q-algebra V with multiplication table (3) will be denoted by V_β. We shall denote by $\mathscr{K}_8$ the set of all algebras with three generators from $\mathfrak{B}_8$. Adjoining to the Q-algebra V_β an arbitrary bilinear form (x, y), for which the form (x, x) is positive definite, we obtain a certain algebra from $\mathscr{K}_8$. Different values of β correspond, of course, to nonisomorphic algebras from $\mathscr{K}_8$. On the other hand, for each individual algebra A we shall obtain a certain set of algebras isomorphic to it, if we subject the basis $A^{(-)}$ to all possible transformations that do not change the multiplication table of the algebra $A^{(-)}$, i.e. the multiplication table (3). The indicated transformations determine the automorphisms of the algebra V_β. Therefore, for the classification of algebras from $\mathscr{K}_8$ up to isomorphism, one must find the group Γ_β of all automorphisms of the algebra V_β. For $\beta = 0$ the group Γ_β coincides with the automorphism group of the Cayley-Dickson algebra; this is the well-known simple Lie group G_2 of dimension 14. The groups Γ_β for $\beta \neq 0$ can also be found. They are all isomorphic to one another and isomorphic to a certain 8-dimensional subgroup of the group Γ_0. Hence one may conclude that the dimension of $\mathscr{K}_8$ is equal to 42.

Introducing on the linear space V_β the scalar product (x, y), for which the basis u_i, v_i, w $(i = 1, 2, 3)$ is orthonormal, we obtain by formula (2) a certain algebra from $\mathscr{K}_8$, which we shall denote by A_β. The algebras A_β for different $\beta \in [0, 1)$ are not only pairwise nonisomorphic, but not even isotopic; this follows from consideration of the determinant $\det L_x$, where L_x is the operator of left multiplication, acting in the linear space A_β by the formula $a \to xa$ (cf. (8)).

4. Let $\mathcal{E}_n$ be the set of all algebras from $\mathfrak{B}_n$ satisfying the additional condition of elasticity, and let A be an arbitrary algebra from $\mathcal{E}_8$. Then in $A^{(-)}$ one can choose a basis $e_1, \dots, e_7$ such that the bilinear form

(x, y) and multiplication in $A^{(-)}$ satisfy the relations (9).

$$(e_i, e_j) = \delta_{ij}, \qquad e_i \cdot e_j = \sum_{k=1}^{7} c_{ij}^k e_k, \quad i, j = 1, \dots, 7, \tag{4}$$

where δ_{ij} is the Kronecker symbol, $c_{ij}^k = -c_{ji}^k = -c_{ik}^j$ $(i, j, k = 1, \dots, 7)$. Relations (4) remain invariant under arbitrary orthogonal transformations of the basis of $A^{(-)}$.

For any element $x \in A^{(-)}$, denote its length by $|x|$: $|x|^2 = (x, x)$. Let H be an arbitrary two-dimensional subspace of $A^{(-)}$, and let $\{x, y\}$ be an arbitrary orthonormal basis of H. Since $A^{(-)}$ is a Q-algebra, in particular $x \cdot y \neq 0$, $|x \cdot y| > 0$. It is easy to see that the number $\alpha = |x \cdot y|$ does not depend on the choice of the orthonormal basis of H; therefore one may put $\alpha = \alpha(H)$, thus obtaining a numerical function on the Grassmann manifold M of two-dimensional subspaces of the space $A^{(-)}$. The function $\alpha(H)$ is continuous on M. Using points at which either the maximum of the function $\alpha(H)$, or its maximum on certain compact submanifolds of M, is attained, we arrive at the construction of an orthonormal basis of $A^{(-)}$ whose multiplication table depends on only 14 parameters. Under certain restrictions of inequality type these parameters are determined uniquely and have an invariant meaning. Hence the following theorem follows:

Theorem 5. The space $\mathcal{E}_8$ has dimension 14.

It is interesting to note that every elastic division algebra of dimension $n > 2$ over R is either commutative or is an algebra with associative powers. But from the result announced in (10) it follows that commutative algebras in D_n for $n > 2$ do not exist. Therefore the following theorem holds:

Theorem 6. If an algebra $A \in D_n$ $(n = 1, 2, 4, 8)$ is elastic, i.e. satisfies the identity

$$(xy)x = x(yx),$$

then A is an algebra with associative powers.

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Note: Figure translations are in progress. See original paper for figures.