DIFFUSION OF RADIATION

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Abstract

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FOR A STRONGLY ELONGATED SCATTERING INDICATRIX

In the theory of radiation diffusion it is usually assumed that the scattering indicatrix is spherical or does not differ greatly from one. The development of the theory under this assumption has by now achieved considerable success. The problem of radiation diffusion in a medium with a strongly elongated scattering indicatrix has been studied to a much lesser extent. Meanwhile, such media are encountered in practice quite often. These include, in particular, dusty galactic nebulae, planetary atmospheres, and water basins.

In the present note an approximate method is proposed for solving the indicated problem, consisting in reducing the integro-differential equation of radiative transfer to a second-order differential equation in partial derivatives.

For simplicity we shall assume that radiation diffusion takes place in a medium consisting of plane-parallel layers. Let $I(\tau, \vartheta, \varphi)$ denote the intensity of radiation traveling at optical depth τ at an angle ϑ to the normal and with azimuth φ. As is known (see, for example, ([1])), the radiative transfer equation has the form

$$\cos\vartheta \, \frac{\partial I(\tau, \vartheta, \varphi)}{\partial \tau} = -I(\tau, \vartheta, \varphi) + \frac{\lambda}{4\pi} \int_0^{2\pi} d\varphi' \int_0^{\pi} x(\gamma) \, I(\tau, \vartheta', \varphi') \sin\vartheta' \, d\vartheta', \quad (1)$$

where $x(\gamma)$ is the scattering indicatrix (γ is the angle between the directions of the scattered and incident rays), and λ is the probability of photon survival in an elementary act of scattering.

We assume that the scattering indicatrix is strongly elongated forward, i.e., the function $x(\gamma)$ has a sharp maximum in the direction characterized by the angles ϑ and φ. Therefore the quantity $I(\tau, \vartheta', \varphi')$ may be approximately replaced by its Taylor expansion in powers of $\vartheta' - \vartheta$ and $\varphi' - \varphi$. Restricting ourselves to the quadratic terms of this expansion, instead of equation (1) we find

$$\cos\vartheta\,\frac{\partial I}{\partial\tau}+(1-\lambda)I=\lambda u\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial I}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 I}{\partial\varphi^2}\right], \tag{2}$$

where

$$u=\frac{1}{8}\int_0^\pi x(\gamma)\sin^3\gamma\,d\gamma. \tag{3}$$

Equation (2) can also be obtained in a somewhat different way. We replace the given scattering indicatrix approximately by the indicatrix

$$x(\gamma)=\begin{cases} c, & \gamma\leqslant\gamma_0,\\ 0, & \gamma>\gamma_0,\end{cases} \tag{4}$$

where the quantities γ_0 and c are related to each other by the normalization condition

$$c(1-\cos\gamma_0)=2. \tag{5}$$

Substituting (4) into (1), expanding $I(\tau,\vartheta',\varphi')$ in a Taylor series and using the smallness of γ_0, instead of equation (1) we again arrive at equation (2), in which

$$u=1/2c. \tag{6}$$

The more strongly elongated the scattering indicatrix, the smaller the angle γ_0 and the larger the quantity c. Usually the elongation of the indicatrix is characterized by the parameter x_1, which is the first term of its expansion in Legendre polynomials, i.e.,

$$x_1=\frac{3}{2}\int_0^\pi x(\gamma)\cos\gamma\sin\gamma\,d\gamma. \tag{7}$$

Finding the value of x_1 for the indicatrix (4) and setting it equal to the value of x_1 for the real indicatrix, we obtain

$$c=3/(3-x_1). \tag{8}$$

Substituting (8) into (6), we have

$$u=(3-x_1)/6. \tag{9}$$

The determination of the parameter u by formula (9) may prove more successful than by formula (3).

Thus, for the approximate determination of the radiation intensity $I(\tau, \vartheta, \varphi)$, we have obtained equation (2), in which the quantity u is determined by formula (3) or (9) (or by some other suitable formula). It is evident that the results of using equation (2) will be more accurate the more scatterings a photon undergoes in the given medium (i.e., the greater the optical thickness of the medium and the closer the quantity λ is to 1). This is explained by the fact that after many scatterings, even with the indicatrix (4), a photon can greatly change its direction, as a result of which photons scattered backward also appear.*

To solve any particular problem, boundary conditions must be added to equation (2). If the radiation sources are outside the medium, then these conditions must determine the intensity of the radiation incident on the boundaries from outside. If the sources are located inside the medium, then the boundary conditions must express the absence of external radiation. In the latter case, a term characterizing the emissive capacity of the medium should be introduced into equation (2).

As an example, let us use equation (2) to solve the problem of the light regime in deep layers of a semi-infinite medium with external radiation sources. This problem can also be solved exactly (see ([1])), and the results of the approximate and exact solutions can be compared with one another.

In this case the radiation intensity does not depend on the azimuth, and it can be represented in the form

$$I(\tau, \vartheta) = y(\vartheta) e^{-k\tau}, \tag{10}$$

where k is a constant depending only on the scattering indicatrix and the value of λ.

Substituting (10) into (2), we obtain the following equation for determining the function $y(\vartheta)$:

$$\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial y}{\partial\vartheta} \right) = (a - b\cos\vartheta) y, \tag{11}$$

where

$$a = (1 - \lambda)/\lambda u, \qquad b = k/\lambda u. \tag{12}$$

* In the theory of particle diffusion, an equation similar to (2) (with $\sin\vartheta$ replaced by ϑ and without the derivative with respect to φ) was used to study small deviations of a particle from its initial direction.

We shall seek the solution of equation (11) in the form of an expansion in Legendre polynomials, i.e., we put

$$y(\vartheta) = \sum_{0}^{\infty} y_n P_n(\vartheta). \tag{13}$$

Substitution of (13) into (11) leads to the following recurrence formula for determining the coefficients y_n:

$$[a + n(n+1)]y_n = b\left(\frac{n}{2n-1}y_{n-1} + \frac{n+1}{2n+3}y_{n+1}\right). \tag{14}$$

From the solvability condition for the system of homogeneous equations (14), consisting in the vanishing of its determinant, one can find the dependence between the quantities a and b. Hence, with the aid of formulas (12), one obtains the dependence between the quantities k, λ, u.

Table 1

b	a	λ ($u =$ 0.05)	k ($u =$ 0.05)	λ ($u =$ 0.10)	k ($u =$ 0.10)	λ ($u =$ 0.15)	k ($u =$ 0.15)
0	0	1	0	1	0	1	0
1	0.16	0.992	0.050	0.984	0.098	0.977	0.146
2	0.56	0.973	0.097	0.947	0.189	0.923	0.277
3	1.09	0.948	0.142	0.902	0.271	0.859	0.387
4	1.70	0.922	0.184	0.854	0.342	0.796	0.478
5	2.37	0.894	0.224	0.809	0.404	0.738	0.554
6	3.06	0.867	0.260	0.766	0.460	0.685	0.617
7	3.78	0.841	0.294	0.726	0.508	0.638	0.670
8	4.52	0.816	0.326	0.689	0.551	0.596	0.715
9	5.28	0.791	0.356	0.655	0.590	0.558	0.753
10	6.05	0.768	0.384	0.623	0.623	0.524	0.787

The determinant of system (14) is equal to

$$\Delta = \begin{vmatrix} a - \frac{1}{3}b & 0 & 0 & \cdots \\ -b & a + 2 - \frac{2}{5}b & 0 & \cdots \\ 0 & -\frac{2}{3}b & a + 6 - \frac{3}{7}b & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}. \tag{15}$$

From the condition $\Delta = 0$, using a well-known device (see, for example, (^{2})), we can express a as a function of b in the form of a continued fraction. Denote by Δ_n the determinant obtained from (15) by deleting the first n rows and columns. Then we have

$$\Delta = a\Delta_1 - \frac{1}{3}b^2\Delta_2, \tag{16}$$

$$\Delta_1 = (a+2)\Delta_2 - \frac{4}{15}b^2\Delta_3 \tag{17}$$

and so on. Since $\Delta = 0$, it follows from (16) that

$$a = \frac{1}{3}b^2\Delta_2/\Delta_1. \tag{18}$$

Substituting into (18) expression (17) and the subsequent expressions for $\Delta_2, \Delta_3, ...$, we obtain

$$a = \frac{1}{3}\,\cfrac{b^2}{a+2-\cfrac{4}{15}\,\cfrac{b^2}{a+6-\cfrac{9}{35}\,\cfrac{b^2}{a+12-\cdots}}}. \tag{19}$$

Formula (19), for small b, gives $a = \frac{1}{6}b^2$. Hence, on the basis of (12) and (9), we find

$$\lambda = 1 - k^2/(3 - x_1). \tag{20}$$

Using (20), from (13) and (14) we obtain

$$y(\vartheta) = y_0\left(1 + \frac{3k}{3 - x_1}\cos\vartheta\right). \tag{21}$$

Formulas (20) and (21) are also obtained from the rigorous theory for small k. This, to some extent, justifies the definition of the parameter u by formula (9).

Table 1 gives the values of a, calculated by formula (19), for values of b from 0 to 10. There, in numerical form, the dependence between the quantities λ and k is also given for three values of the parameter u.

After the dependence between a and b has been found, the radiation intensity can be determined from formulas (13) and (14) (up to the constant factor y_0).

Let us apply the results obtained to the case of the Henyey—Greenstein scattering indicatrix [3]:

$$x(\gamma) = (1-g^2)/(1+g^2-2g\cos\gamma)^{3/2}, \tag{22}$$

Fig. 1

Figure 1: Fig. 1

which has often been used by astrophysicists. Since for it $x_1 = 3g$, on the basis of formulas (5) and (7) we have

$$u = \frac{1}{2}(1 - g). \tag{23}$$

This means that the approximate dependence between the quantities λ and k presented in the table corresponds to the indicatrix (22) for $g = 0.7, 0.8, 0.9$.

Fig. 1

In Fig. 1, for the indicatrix (22), two curves are given, one of which represents the exact dependence between λ and k (for $g = 0.8$), and the other the approximate one (for $u = 0.1$). We see that, for values of λ close to 1, both curves are close to each other. Consequently, the use of equation (2) leads in this case to satisfactory results.

One may expect that similar results will also be obtained when solving other problems by means of equation (2).

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