STABILITY OF THE MOTION OF A CHARGED PARTICLE IN THE SIMPLEST GEOMETRY OF A MODULATED MAGNETIC FIELD

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STABILITY OF THE MOTION OF A CHARGED PARTICLE IN THE SIMPLEST GEOMETRY OF A MODULATED MAGNETIC FIELD

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1. For a number of problems in the physics of laboratory and cosmic plasma ([1–4]), the character of the motion of a charged particle in magnetic fields modulated in time is of interest. This problem was first considered by L. I. Rudakov and R. Z. Sagdeev ([1]) for the simplest geometry of a homogeneous magnetic field in a system with cylindrical symmetry:

$$\mathbf{H} = H_0(1 - h\cos\gamma t)\mathbf{z}, \tag{1}$$

where $\mathbf{z}$ is a unit vector along the z-axis; H_0 is the amplitude of the constant magnetic field; γ is the modulation frequency and $h \equiv H_m/H_0$ is the modulation depth (H_m is the amplitude of the alternating magnetic field). In ([1]) an additional simplifying assumption was introduced concerning a small modulation depth of the alternating magnetic field relative to the strength of the constant field ($h \ll 1$).

A number of works have considered the behavior of a charged particle in a magnetic field increasing linearly with time ([2]), varying sawtooth-like in time ([3]), and also having the character of a single pulse ([4]). The extensive investigations by V. P. Silin et al. ([5]) on potential oscillations of a plasma in a high-frequency electric field pertain to a fundamentally different problem.

2. We shall solve the problem of the stability of the motion of a charged particle in the magnetic field (1). In the presence of cylindrical symmetry, there corresponds to it an induction electric field proportional to the radius-vector $\mathbf{r}$:

$$\mathbf{E} = h\frac{H_0\gamma}{2c}\sin\gamma t\,[\mathbf{r}, \mathbf{z}]. \tag{2}$$

In real systems the homogeneity of the magnetic field is usually disturbed because of boundary conditions at the periphery of the cylinder. Therefore our results are applicable to the region near the cylinder axis. In this case the motion of a particle with charge Ze and mass M is described by the equation:

$$\frac{d^2\mathbf{r}}{dt^2} = h\frac{\gamma\omega_c}{2}\sin\gamma t\,[\mathbf{r},\mathbf{z}] + \omega_c(1-h\cos\gamma t)[\mathbf{v},\mathbf{z}], \tag{3}$$

where $\omega_c \equiv ZeH_0/Mc$ is the cyclotron frequency.

By the substitution

$$w \equiv (x+iy)\exp\left(\frac{i\omega_c}{2}\int_0^t (1-h\cos\gamma t)\,dt\right)$$

equation (3) is reduced to an equation of Hill type

$$\frac{d^2w}{dt^2} + \frac{\omega_c^2}{4}(1-h\cos\gamma t)^2 w = 0. \tag{4}$$

An equation of Hill type is obtained as a consequence of the fact that the electric-field strength is proportional to the distance from the axis of the cylinder, which is true only for the simplest field geometry under consideration. If the modulation depth $h = H_m/H_0 \ll 1$, then equation (4) becomes the Mathieu equation

$$\frac{d^2w}{dt^2} + \frac{\omega_c^2}{4}(1-2h\cos\gamma t)w = 0,$$

whose stability diagram (the Ince–Strutt diagram) is well known ([6]). This limiting case was considered in ([1]). We shall solve the problem for an arbitrary modulation depth h and obtain results applicable, in particular, to the limiting case $h \to \infty$.

3. In the case of fields (1) and (2), when the stability of the particle motion is determined by the Hill equation (4) (since $|\mathbf{r}|^2 = x^2 + y^2 = |w|^2$), the stability diagram can be obtained as follows. From the general theory of the Hill equation

$$d^2w/d\xi^2 + [a - 2q\psi(\xi)]w = 0 \tag{5}$$

it is known ([6]) that unstable solutions correspond to those points in the (a, q) plane for which the real part of the characteristic exponent μ is different from zero. Here $\psi(\xi)$ is a periodic function of ξ with period T, and a and q are parameters. For equation (4), $a \equiv (\omega_c/\gamma)^2$ and $q \equiv ah = h(\omega_c/\gamma)^2$. As shown in ([7]), the characteristic exponent μ for Hill' s equation can be found from the equation (for $\mu \neq 0$):

$$\rho^2 - \rho\left[\frac{w(0)+w(2T)}{w(T)}\right] + 1 = 0, \tag{6}$$

where $\rho \equiv e^{\mu T}$, and $w(0)$, $w(T)$, $w(2T)$ are the values of the function $w(\xi)$, satisfying equation (5), at $\xi = 0$, $\xi = T$, and $\xi = 2T$, respectively. Equation (6) is solved numerically together with equation (5) for various values of the parameters a and q; for the function $w(\xi)$ and $dw/d\xi$ one may prescribe arbitrary initial values, provided only that $w(0)$ and $dw/d\xi|_{\xi=0}$ do not vanish simultaneously.

By this method, with the aid of an electronic computer, we constructed the stability diagram for equation (4), shown in Fig. 1. Since $h = q/a$, the locus of points satisfying equation (4) at a constant value of h is a ray issuing from the origin at an angle $\alpha = \operatorname{arcctg} h$ to the axis of abscissas. As $h \to 0$, the corresponding ray approaches the ordinate axis, and the stability diagram shown in Fig. 1 becomes the Ince–Strutt diagram. In this case we obtain the usual parametric resonances considered in ([1]). For $h \gtrsim 1$, the stability diagram for Hill' s equation (4) differs from the Ince–Strutt diagram by a much smaller width of the instability zones.

4. For the limiting case of large modulation depths ($h \to \infty$, i.e. $H_0 \to 0$), equation (4) becomes

$$\frac{d^2w}{dt^2} + \frac{\omega_c^2 h^2}{4}\cos^2\gamma t \cdot w = 0$$

or, replacing $\cos^2\gamma t$ by $\frac{1}{2}(1+\cos 2\gamma t)$,

$$\frac{d^2w}{dt^2} + \frac{\omega_c^2 h^2}{8}(1+\cos 2\gamma t)\cdot w = 0. \tag{7}$$

Thus, as $h \to \infty$, we obtain the Mathieu equation with the natural frequency increased by a factor of $h/\sqrt{2}$, modulation depth equal to unity, and modulation frequency doubled. It follows that, with increasing modulation depth, the instability zones shift toward the

high modulation frequencies or low natural frequencies. Comparing equation (7) with the canonical form of Mathieu' s equation (6)

$$\frac{d^2w}{d\xi^2} + [a_1 - 2q_1\cos 2\xi]w = 0,$$

we conclude that the limiting form of the instability zones as $h \to \infty$ can be found by drawing on the Ince-Strutt diagram the ray corresponding to $2q_1 = -a_1$ (which, because of the symmetry of the diagram, does not differ from $2q_1 = a_1$), and setting

Fig. 1 and Fig. 2: diagrams of the first three instability zones

Figure 1: Fig. 1 and Fig. 2: diagrams of the first three instability zones

$$a_1 = 2q_1 = \omega_c^2 h^2/8\gamma^2 = ah^2/8. \tag{8}$$

Fig. 1. Diagram of the first three instability zones (hatched) of equation (4) in coordinates (a, q)

Fig. 2. Diagram of the first three instability zones (hatched) of equation (4) in coordinates (b, p)

For large modulation depths the stability diagram in the traditional coordinates (a, q) is inconvenient, since the zones converge to the origin. Here new coordinates are more convenient:

$$b \equiv \sqrt{a} = \omega_c/\gamma, \qquad p \equiv hb.$$

For large modulation depths, according to relation (8),

$$b = \sqrt{8a_1}/h, \qquad p = \sqrt{8a_1}.$$

As $h \to \infty$, the parameter p remains finite. The stability diagram in coordinates (b, p) is shown in Fig. 2. On the abscissa axis are plotted the zone boundaries for $h \to \infty$, found from the Ince-Strutt graph by means of relation (8). The diagram shows that the results of the numerical calculations pass smoothly into the limiting values.

It follows from the foregoing that, for moderate and large modulation depths, there exist rather wide instability zones in which the transverse energy of the particle increases. In the problem under consideration, with a magnetic field homogeneous in space, the particle can obtain energy only from the induction electric field, i.e., only owing to the nonadiabaticity of the motion. Energy transfer can be appreciable only under strong nonadiabaticity, i.e., at modulation frequencies of the order of, or higher than, the natural frequency. At low modulation frequencies, i.e., at higher parametric resonances, instability does formally exist, but energy transfer occurs extremely slowly, since

the real part of the characteristic exponent is small. From the results presented it is clear that the condition of nonadiabaticity becomes the more stringent, the greater the relative depth of modulation. For large values of h, the lower parametric resonances require modulation frequencies approximately h times greater than the natural (cyclotron) frequency. The most effective parametric resonance should occur at $h \sim 1$, when the instability zones are already sufficiently wide, but instability is still possible at not too high modulation frequencies.

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Note: Figure translations are in progress. See original paper for figures.