ON SOME EXTENSIONS OF TOPOLOGICAL VECTOR SPACES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.88 : 513.83

MATHEMATICS

V. D. GOLOVIN

ON SOME EXTENSIONS OF TOPOLOGICAL VECTOR SPACES

(Presented by Academician S. N. Bernshtein, 21 VI 1966)

1. Let E be a separable locally convex topological vector space over the field R of real numbers, and let E' be the vector space conjugate to E, endowed with the strong topology. Let A be an arbitrary nonempty set and let $\mathfrak{B}(A)$ be the Banach algebra of all bounded real-valued numerical functions on A. Define the $\mathfrak{B}(A)$-module

$$E_{(A)} = L_R(E', \mathfrak{B}(A))$$

of all continuous linear mappings of the topological vector space E' into the Banach space $\mathfrak{B}(A)$. Considering the module $E_{(A)}$ as a vector space over R, define the linear mapping

$$i : E \to E_{(A)},$$

which assigns to each element $s \in E$ the continuous linear mapping $x : E' \to \mathfrak{B}(A)$ such that $x(t)$, for every $t \in E'$, is the function on A identically equal to $t(s)$. It is immediately evident that the mapping i is injective; therefore $E_{(A)}$ may be regarded as a module obtained from the topological vector space E by extending its field of scalars to the algebra $\mathfrak{B}(A)$.

Analogously one may define the $\mathfrak{B}(A)$-module

$$E'_{(A)} = L_R(E, \mathfrak{B}(A))$$

of all continuous linear mappings $E \to \mathfrak{B}(A)$ and the linear mapping

$$j : E' \to E'_{(A)},$$

which assigns to each element $t \in E'$ the continuous linear mapping $y : E \to \mathfrak{B}(A)$ such that $y(s)$, for every $s \in E$, is the function on A identically equal to $t(s)$. Since the mapping j is injective, $E'_{(A)}$ may be regarded as a module obtained from the space E' by extending its field of scalars to the algebra $\mathfrak{B}(A)$.

2. Let ε_α, for each $\alpha \in A$, be the continuous linear form on $\mathfrak{B}(A)$ such that $\varepsilon_\alpha(b) = b(\alpha)$ for every $b \in \mathfrak{B}(A)$. Then each element $x \in E_{(A)}$ is identified with the equicontinuous family of linear forms $x_\alpha = \varepsilon_\alpha \circ x$ $(\alpha \in A)$ on the topological vector space E'; analogously, each element $y \in E'_{(A)}$ is identified with the equicontinuous family of linear forms $y_\alpha = \varepsilon_\alpha \circ y$ $(\alpha \in A)$ on E. Since every equicontinuous set in E' is bounded in the strong topology, the values $\langle x_\alpha, y_\beta \rangle$ $(\alpha, \beta \in A)$ of the canonical bilinear form defined on the product $E'' \times E'$ are bounded in the aggregate. We define on the product $E_{(A)} \times E'_{(A)}$ a $\mathfrak{B}(A)$-bilinear form by assigning to each pair $(x, y) \in E_{(A)} \times E'_{(A)}$ the bounded numerical function $\alpha \to \langle x_\alpha, y_\alpha \rangle$ on A, which we shall denote by $\langle x, y \rangle$. We shall show that the $\mathfrak{B}(A)$-bilinear form so defined,

$$(x, y) \to \langle x, y \rangle$$

puts the modules $E_{(A)}$ and $E'_{(A)}$ in duality. Indeed—

therefore, if $\langle x, y \rangle = 0$ for some $x \in \dot{E}_{(A)}$ and every $y \in E'_{(A)}$, then $\langle x_\alpha, t \rangle = 0$ $(\alpha \in A;\ t \in E')$, whence $x_\alpha = 0$ $(\alpha \in A)$, i.e. $x = 0$. If $\langle x, y \rangle = 0$ for every $x \in E_{(A)}$ and some $y \in E'_{(A)}$, then $\langle s, y_\alpha \rangle = 0$ $(\alpha \in A;\ s \in E)$ and, consequently, $y = 0$. Finally, it is obvious that $\varepsilon_\alpha(\langle i(s), j(t) \rangle) = \langle s, t \rangle$ $(\alpha \in A)$ for every pair $(s, t) \in E \times E'$. Thus the following has been proved.

Theorem 1. *On the product $E_{(A)} \times E'_{(A)}$ there exists a canonical $\mathfrak{B}(A)$-bilinear form $(x, y) \mapsto \langle x, y \rangle$, putting the modules $E_{(A)}$ and $E'_{(A)}$ in duality and such that the value $\langle i(s), j(t) \rangle$ for each pair $(s, t) \in E \times E'$ identically coincides with the value $\langle s, t \rangle$ of the canonical bilinear form putting the spaces E and E' in duality.*

3. By the weak topology $\sigma(E_{(A)}, E'_{(A)})$ we shall mean the weakest of the topologies in $E_{(A)}$ for which all $\mathfrak{B}(A)$-linear forms of the form $x \mapsto \langle x, y \rangle$ $(y \in E'_{(A)})$ are continuous. This is a topology consistent with the structure of the $\mathfrak{B}(A)$-module in $E_{(A)}$, for which a fundamental system of neighborhoods of zero is formed by the sets

$$W(y_1, \dots, y_n; \varepsilon) = \{x \in E_{(A)} :\ \|\langle x, y_k \rangle\| \leqslant \varepsilon\ (1 \leqslant k \leqslant n)\},$$

each of which is determined by a finite set of elements $y_k \in E'_{(A)}$ $(1 \leqslant k \leqslant n)$ and a number $\varepsilon > 0$.

Theorem 2. *The module $L_{\mathfrak{B}(A)}(E_{(A)}, \mathfrak{B}(A))$ of all $\mathfrak{B}(A)$-linear forms on $E_{(A)}$, continuous in the weak topology $\sigma(E_{(A)}, E'_{(A)})$, is canonically identifiable with the module $E'_{(A)}$: every continuous $\mathfrak{B}(A)$-linear form on $E_{(A)}$ is uniquely representable in the form $x \mapsto \langle x, y \rangle$, where $y \in E'_{(A)}$.*

Indeed, let g be an arbitrary continuous $\mathfrak{B}(A)$-linear form on $E_{(A)}$. Then $g(bx) = bg(x)$, whatever $b \in \mathfrak{B}(A)$ and $x \in E_{(A)}$; consequently, $g_\alpha(bx) = b(\alpha)g_\alpha(x)$, where $g_\alpha = \varepsilon_\alpha \circ g$ $(\alpha \in A)$. In particular, $g_\alpha(x) = 0$ if $x_\alpha = 0$; in other words, for each $\alpha \in A$ there is defined a mapping $x_\alpha \mapsto g_\alpha(x)$, which is a linear form on the vector space E'', continuous in the topology $\sigma(E'', E')$. Therefore $g_\alpha(x) = \langle x_\alpha, y_\alpha \rangle$, where $y_\alpha \in E'$ $(\alpha \in A)$. On the other hand, by virtue of the continuity of the form g in the topology $\sigma(E_{(A)}, E'_{(A)})$ there exist $z_k \in E'_{(A)}$ $(1 \leqslant k \leqslant n)$ such that $|\langle x_\alpha, y_\alpha \rangle| \leqslant 1$ $(\alpha \in A)$ if $\|\langle x, z_k \rangle\| \leqslant 1$ $(1 \leqslant k \leqslant n)$. Since the family of linear forms $z_{k\alpha} = \varepsilon_\alpha \circ z_k$ $(1 \leqslant k \leqslant n;\ \alpha \in A)$ is equicontinuous on E, for every s from some neighborhood of zero in E one has $|\langle s, z_{k\alpha} \rangle| \leqslant 1$ $(1 \leqslant k \leqslant n;\ \alpha \in A)$, whence $|\langle s, y_\alpha \rangle| \leqslant 1$ $(\alpha \in A)$. Thus the family (y_α) is equicontinuous on E and, consequently, defines some element $y \in E'_{(A)}$, for which $\varepsilon_\alpha \circ y = y_\alpha$ $(\alpha \in A)$ and $g(x) = \langle x, y \rangle$ $(x \in E_{(A)})$. If $g(x) = \langle x, z \rangle$ $(x \in E_{(A)})$ for some z in $E'_{(A)}$, then, in view of the duality between $E_{(A)}$ and $E'_{(A)}$, from $\langle x, y - z \rangle = 0$ $(x \in E_{(A)})$ it follows that $y = z$, which completes the proof of the theorem.

By analogy with the preceding, endow the module $E'_{(A)}$ with the weak topology $\sigma(E'_{(A)}, E_{(A)})$, i.e. the weakest of the topologies for which all $\mathfrak{B}(A)$-linear forms of the form $y \mapsto \langle x, y \rangle$ $(x \in E_{(A)})$ are continuous. Then the following holds.

Theorem 3. *The module $L_{\mathfrak{B}(A)}(E'_{(A)}, \mathfrak{B}(A))$ of all continuous $\mathfrak{B}(A)$-linear forms on $E'_{(A)}$ is canonically identifiable with the module $E_{(A)}$: every continuous $\mathfrak{B}(A)$-linear form on $E'_{(A)}$ is uniquely representable in the form $y \mapsto \langle x, y \rangle$, where $x \in E_{(A)}$.*

4. Whatever the nonempty set A, the linear mapping $i : E \to E_{(A)}$ is continuous if E is endowed with the original topology, and $E_{(A)}$ with the weak topology $\sigma(E_{(A)}, E'_{(A)})$.

Theorem 4. *Whatever the separated locally convex space E, there exists a set A such that the mapping i is a monomorphism.*

Indeed, let U be an arbitrary closed, balanced, convex neighborhood of zero in E; then $U = M^0$, where M is some equicontinuous set in E'. We choose the set A so that its cardinality majorizes the cardinality of any equicontinuous set in E'. In this case $\operatorname{Card}(M) \leq \operatorname{Card}(A)$, and the set M can be represented parametrically as an equicontinuous family of linear forms y_α $(\alpha \in A)$ on E. The family (y_α) determines an element $y \in E'_{(A)}$ such that $\varepsilon_\alpha \circ y = y_\alpha$ $(\alpha \in A)$. Let s be an element in E such that $i(s) \in W(y; 1)$. Then $|\langle s, y_\alpha \rangle| \leq 1$ $(\alpha \in A)$, and conversely. In other words,

$$i(U) = W(y; 1) \cap i(E),$$

and, consequently, i is an isomorphism of the space E onto $i(E)$.

The linear mapping $j : E' \to E$ is continuous if the space E' is endowed with the strong topology, and $E'_{(A)}$ with the weak topology $\sigma(E'_{(A)}, E_{(A)})$.

Analogously to Theorem 4, one proves

Theorem 5. *Whatever the separated locally convex space E, there exists a set A such that the mapping j is a monomorphism.*

5. Let $f : E \to F$ be a continuous linear mapping of the space E into the separated locally convex space F. Then a $\mathfrak{B}(A)$-linear mapping may be defined
$$f_{(A)} : E_{(A)} \to F_{(A)},$$
called associated with f, and assigning to each continuous linear mapping $x : E' \to \mathfrak{B}(A)$ the composition $x \circ f$, which is a continuous linear mapping of the space F' into $\mathfrak{B}(A)$.

Analogously, a $\mathfrak{B}(A)$-linear mapping may be defined

$$^{t}f_{(A)} : F'_{(A)} \to E'_{(A)},$$

called associated with the mapping $^{t}f : F' \to E'$, conjugate to f, and assigning to each continuous linear mapping $y : F \to \mathfrak{B}(A)$ the composition $y \circ f$, which is a continuous linear mapping of the space E into $\mathfrak{B}(A)$.

It follows directly from these definitions that the mapping $^{t}f_{(A)}$ is conjugate to $f_{(A)}$ with respect to the canonical $\mathfrak{B}(A)$-bilinear forms defined respectively on the products $F_{(A)} \times F'_{(A)}$ and $E_{(A)} \times E'_{(A)}$:

$$\langle f_{(A)}(x), y' \rangle = \langle x, {}^{t}f_{(A)}(y') \rangle,$$

whatever $x \in E_{(A)}$ and $y' \in F'_{(A)}$. In particular, the mappings $f_{(A)}$ and $^{t}f_{(A)}$ are continuous in the weak topologies.

Theorem 6. *Let $f : E \to F$ be a continuous linear mapping. In order that the continuous $\mathfrak{B}(A)$-linear mapping $\xi : E_{(A)} \to F_{(A)}$ enter into the commutative diagram*

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "i"'] & F \arrow[d, "i"] \\
E_{(A)} \arrow[r, "\xi"] & F_{(A)}
\end{tikzcd}$$

it is necessary and sufficient that ξ coincide with $f_{(A)}$.

Indeed, the diagram holds for the mapping $f_{(A)}$ associated with f; if it also holds for some continuous $\mathfrak{B}(A)$-linear mapping ξ, then $(f_{(A)} - \xi) \circ i = 0$. Therefore

$$g \circ (f_{(A)} - \xi) = 0$$

for any $\mathfrak{B}(A)$-linear form g on $F_{(A)}$, continuous in the weak topology $\sigma(F_{(A)}, F'_{(A)})$; consequently, $f_{(A)} - \xi = 0$.

Theorem 7. In order that the continuous $\mathcal{B}(A)$-linear mapping $\xi :\ F'_{(A)} \to E'_{(A)}$ enter into the commutative diagram

$$\begin{tikzcd} F' \arrow[r, "{}^t f"] \arrow[d, "j"'] & E' \arrow[d, "j"] \\ F'_{(A)} \arrow[r, "\xi"] & E'_{(A)} \end{tikzcd}$$

it is necessary and sufficient that ξ coincide with ${}^t f_{(A)}$.

6. Let $\sigma(E, E'_{(A)})$ be the weakest of the topologies in E for which all linear mappings $E \to \mathcal{B}(A)$ belonging to $E'_{(A)}$ are continuous. If $A \subset A'$, then the topology $\sigma(E, E'_{(A')})$ is weaker than $\sigma(E, E'_{(A)})$. It can be shown that every linear mapping $f : E \to F$, continuous for the topologies $\sigma(E, E'_{(A')})$, $\sigma(F, F'_{(A')})$, is also continuous for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$. In particular, if f is continuous for the original topologies in E and F, then it is continuous also for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$, whatever the nonempty set A may be.

Theorem 8. Let $f : E \to F$ be a linear mapping, continuous for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$. In order that f be a monomorphism for the same topologies, it is necessary and sufficient that the mapping

$${}^t f_{(A)} : \ F'_{(A)} \to E'_{(A)},$$

associated with ${}^t f$, be surjective.

Indeed, if f is a monomorphism, then, by the Hahn-Banach theorem, every continuous linear mapping $x : E \to \mathcal{B}(A)$ can be represented in the form $x = y \circ f$, where $y \in F'_{(A)}$; hence the mapping ${}^t f_{(A)}$ is surjective. Conversely, if ${}^t f_{(A)}$ is surjective, then to every $x \in E'_{(A)}$ there corresponds a $y \in F'_{(A)}$ such that $x = y \circ f$; consequently, f is injective. Since, moreover, $\langle s, \varepsilon_\alpha \circ x \rangle = \langle f(s), \varepsilon_\alpha \circ y \rangle$ $(s \in E;\ \alpha \in A)$, it follows that f is a monomorphism.

Theorem 9. Let $A \subset A'$. If the mapping $f : E \to F$ is a monomorphism for the topologies $\sigma(E, E'_{(A')})$, $\sigma(F, F'_{(A')})$, then f is a monomorphism also for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$.

Indeed, f is continuous for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$. Every continuous linear mapping $x : E \to \mathcal{B}(A)$ can be represented in the form $x = \theta \circ x'$, where $\theta : \mathcal{B}(A') \to \mathcal{B}(A)$ is the restriction mapping, and $x' : E \to \mathcal{B}(A')$ is some continuous linear mapping. Since f is a monomorphism for the topologies $\sigma(E, E'_{(A')})$, $\sigma(F, F'_{(A')})$, we have $x' = y' \circ f$, where $y' \in F'_{(A')}$. Then $x = y \circ f$, where $y = \theta \circ y'$ is a continuous linear mapping $F \to \mathcal{B}(A)$.

Corollary 1. If the mapping $f : E \to F$ is a monomorphism for the original topologies, then it is a monomorphism also for the topologies $\sigma(E, E'_{(A)})$, $\sigma(F, F'_{(A)})$, whatever the nonempty set A may be.

Indeed, by Theorem 4, for some $A' \supset A$ the topologies $\sigma(E, E'_{(A')})$, $\sigma(F, F'_{(A')})$ coincide with the original topologies in E and F, respectively.

Corollary 2. In order that a continuous linear mapping $f : E \to F$ be a monomorphism (for the original topologies), it is necessary and sufficient that, for every nonempty set A, the mapping

$$^t f_{(A)} : \; F'_{(A)} \to E'_{(A)}$$

be surjective.

Kharkov State University
named after A. M. Gorky

Received
16 VI 1966

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.