Topological classification of singular points and generalized Ljapunov functions

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Abstract

Full Text

Preamble

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TOPOLOGICAL CLASSIFICATION OF SINGULAR POINTS AND GENERALIZED LYAPUNOV FUNCTIONS

In the qualitative theory of differential equations, the problem of the topological classification of singular points remains a fundamental area of research. This study investigates the relationship between the local structure of trajectories in the neighborhood of a singular point and the existence of generalized Lyapunov functions.

Consider a system of differential equations:

$$\frac{dx}{dt} = f(x)$$

where $x \in R^n$ and $f(0) = 0$. We assume that the vector field $f(x)$ satisfies conditions ensuring the existence and uniqueness of solutions in some neighborhood of the origin. The primary objective is to categorize the topological behavior of these solutions by constructing scalar functions that satisfy specific monotonicity properties along the trajectories.

1. Topological Equivalence and Lyapunov Functions

Two singular points are considered topologically equivalent if there exists a homeomorphism of their neighborhoods that maps the trajectories of one system onto the trajectories of the other while preserving their orientation. The classical

Lyapunov stability theory provides a foundation for this classification, but it often requires extension to "generalized" functions to account for non-smooth or complex topological structures.

A generalized Lyapunov function $V(x)$ is defined such that its derivative along the trajectories of the system, denoted by $\dot{V}(x)$, maintains a definite sign in a deleted neighborhood of the singular point. Unlike classical approaches, we do not strictly require $V(x)$ to be continuously differentiable everywhere; instead, we consider functions that are continuous and possess directional derivatives or satisfy Lipschitz conditions.

2. Classification Criteria

The classification of singular points—such as nodes, foci, saddles, and more complex centers—can be rigorously established by examining the level sets of these generalized functions. For instance, the existence of a positive definite function $V(x)$ with a negative definite derivative $\dot{V}(x)$ characterizes an asymptotically stable singular point. Conversely, the presence of regions where $V(x)$ and $\dot{V}(x)$ share the same sign indicates instability, often associated with saddle-type behavior.

[Figure 1: see original paper]

The topological structure is further refined by considering the boundaries of the regions where the generalized Lyapunov function is monotonic. These boundaries correspond to the stable and unstable manifolds (or their generalizations) that partition the phase space. By analyzing the intersection of these manifolds with local

The present work aims to summarize

The results of the study of singular points allow us to propose their classification. All singular points of the system are divided into two types: topologically stable and topologically unstable. We shall call a singular point stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that the condition $\rho(x, x_0) < \delta$ implies that the entire trajectory $f(x, t)$, for $-\infty < t < +\infty$, remains within $S(x_0, \epsilon)$, i.e., in the ϵ-neighborhood of the point x_0. All singular points that are not stable in the sense defined above will be called unstable. A simple logical negation allows for a positive definition of an unstable singular point: a singular point is unstable if there exists some $\epsilon > 0$ such that, no matter how small we choose $\delta(\epsilon) < \epsilon$ in the ball $S(x_0, \delta)$, there exists a point x such that either the positive semi-trajectory $f^+(x, t)$ or the negative semi-trajectory $f^-(x, t)$ has points outside $S(x_0, \epsilon)$. Note that these definitions are not linked to the finite dimensionality of the space or its topological nature and can be applied to any metric space.

The topological non-equivalence of stable and unstable singular points in a locally compact metric space is easily established. Indeed, let x_0 be an unstable singular point. Let $\epsilon > 0$ be the value whose existence follows from the definition of instability. Consider $S(x_0, \epsilon)$. Assume for the sake of contradiction that there exists a topological mapping of some $S(x_0, \delta)$, where $\delta \leq \epsilon$, onto an ϵ-neighborhood of a stable point y_0, such that trajectories are mapped to trajectories. Let y_0 be stable; then, on one hand, by the definition of a stable point, there exists some η such that the bundle of trajectories $f(S(y_0, \eta); -\infty, +\infty)$ is contained within $S(y_0, \epsilon)$ and, consequently, every trajectory of this bundle has its α- and ω-limit points in $S(y_0, \epsilon)$. Since the mapping is topological, their pre-images must have α- and ω-limit points in the pre-image of $S(y_0, \epsilon)$, and even more so in the pre-image of $S(y_0, \eta)$. Meanwhile, in the pre-image of $S(y_0, \eta)$, due to instability, there must be initial points of trajectories whose positive or negative semi-trajectories leave $S(x_0, \epsilon)$ and thus leave the pre-image of $S(y_0, \epsilon)$. The resulting contradiction proves the topological non-equivalence of stable and unstable points.

V. NEMYTSKII. Let us first consider unstable points. Suppose there exists an $\epsilon > 0$ and a sufficiently small $\delta(\epsilon)$ such that in the ball $S(x_0, \delta)$ there are only points such that the positive semi-trajectory $f(x, t)$ has points outside $S(x_0, \epsilon)$, while the negative semi-trajectory of every point in $S(x_0, \delta)$ is immersed in $S(x_0, \epsilon)$, or, conversely, only the negative semi-trajectory leaves $S(x_0, \epsilon)$. Unstable points possessing these properties will be called negative (respectively, positive) generalized nodes. Generalized nodes can be points with very complex structures, since any neighborhood of such a point may contain recurrent motions; as shown by Yu. S. Bogdanov [**?**], it may even happen that a generalized node is not a dynamical limit point for any trajectory.

Examples of the first type are easily constructed using a building element we call a "twisted cylinder." Every trajectory, except for the axial one, remains for all $t > 0$ on a cylinder C resting on a circle $L = C$, and has this circle as its ω-limit set. However, for the continuity of the field, we require that as the radius decreases (approaching zero), the trajectory turns into a helix. Thus, in any neighborhood of x_0, there are periodic solutions. The example of the phenomenon indicated by Yu. S. Bogdanov is significantly more complex.

Unstable singular points that are not generalized nodes will be called generalized saddles. We provide a positive characterization of a generalized saddle.
Theorem 1. For x_0 to be a generalized saddle, it is necessary and sufficient that there exists some $\eta > 0$ such that any neighborhood $S(x_0, \eta)$ contains both α- and ω-limit points of trajectories starting on the boundary of $S(x_0, \eta)$. Sufficiency follows directly from the definition of an unstable point. For the value $\eta > 0$ in the definition, one can take any ϵ. Necessity can be established by the reasoning of Yu. S. Bogdanov [**?**], applied twice: once considering that x_0 is Lyapunov unstable in the positive direction, and a second time considering that it is "Lyapunov unstable in the negative direction" (i.e., when t is replaced by $-t$).

An example of a generalized saddle that no trajectory enters and which is neither an α- nor an ω-limit point of any trajectory can be constructed as follows.

Suppose we are given the surface of a torus filled with periodic trajectories along the meridians, and let this torus be the ω-limit set of a family of trajectories located outside the torus. We perform a continuous and one-to-one deformation of the field everywhere except for one meridian, such that this marked meridian is collapsed into a point, and the torus is transformed into a topological ball where two points are identified, creating a singular point at this double point. The interior of the pinched torus is homeomorphic to the interior of a topological ball and is filled with spirals approximating the surface in the positive sense. Such a filling can be visualized, for example, by considering a ball with marked north and south poles, where the field provides rotation around the axis while the plane rotates around the axis. After this construction, we identify the north and south poles. This example shows that the "saddle whiskers" can be filled with periodic solutions. Note that the idea for this example was communicated to the author by L. E. Reizin. From the general class of unstable singular points, we distinguish the class of simple unstable singular points.

Definition. A point x_0 is called a simple unstable singular point if there exists an $\epsilon_0 > 0$ such that all trajectories starting in the neighborhood $S(x_0, \epsilon)$, where $\epsilon < \epsilon_0$, either have no dynamical limit points in this neighborhood at all, or their dynamical limit point coincides with x_0. Every neighborhood $S(x_0, \epsilon)$ possessing the property specified in the definition will be called a small neighborhood of the point x_0. Consequently, if we fix any small neighborhood, any trajectory starting in $S(x_0, \epsilon)$ will be one of three types: 1. **Hyperbolic**: having no dynamical limit points and leaving $S(x_0, \epsilon)$ in a finite time interval in both directions. 2. **Parabolic (negative)**: having x_0 as its α-limit point and leaving in the positive direction; **parabolic (positive)**: having x_0 as its ω-limit point and leaving in the negative direction. 3. **Elliptic**: having x_0 as both its α- and ω-limit point.

Remark. If there is at least one elliptic trajectory, the point x_0 is unstable. Indeed, let the maximum deviation of a point moving along an elliptic trajectory be ϵ_1; consider a neighborhood $S(x_0, \epsilon)$ where $\epsilon < \epsilon_1$. The trajectories filling this neighborhood will not satisfy the stability condition.

Theorem 2. There exists no singular point having a neighborhood filled only with hyperbolic curves relative to $S(x_0, \epsilon)$. Indeed, let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence of points converging to x_0 through which hyperbolic curves pass, and let $y_1, y_2, \ldots, y_n, \ldots$ be the points of final exit (i.e., points in whose immediate vicinity on L there are points not belonging to $S(x_0, \epsilon)$). The time lengths of the arcs $x_n y_n$ tend to infinity; consequently, through the limit point for y_n, there passes a trajectory having α-limit points at x_0, which is therefore not hyperbolic.

It is interesting to determine whether some neighborhood of a saddle point can be filled only with elliptic trajectories. Let us clarify the meaning of this statement: we say that a point is purely elliptic if there exists some $\eta < \epsilon$ such that all trajectories starting in $S(x_0, \eta)$ are elliptic trajectories entirely immersed in $S(x_0, \epsilon)$. **Hypothesis.** Purely elliptic points cannot exist. **Remark.** If a point is simple and does not contain elliptic trajectories in $S(x_0, \epsilon)$, the above reasoning shows that in the neighborhood of every saddle point there are "separatrix sets" filled with positive parabolic trajectories and a "separatrix set" filled with negative parabolic curves. In the general case, as shown by the previous example, these separatrix sets may be filled with closed curves, i.e., trajectories that do not enter the singular point. We note that while the author possessed both the formulation and proof of Theorem 2, M. A. Krasnoselskii kindly provided me with a manuscript by E. Mukhamadiev, which contained a similar theorem.

Example. Let a dynamical system consist of a family of parallel lines. On one of these lines, a point is marked and declared singular. In this system, all curves are hyperbolic, except for one α-curve and one ω-curve.

V. NEMYTSKII. **Theorem 3.** If a singular point x_0 is simple and there are no elliptic trajectories in some small neighborhood $S(x_0, \eta)$, then no matter how small $S(x_0, \epsilon) \subset S(x_0, \eta)$ is, there are points in $S(x_0, \epsilon)$ that leave $S(x_0, \eta)$ in both directions. Let L_ω be a trajectory from $S(x_0, \eta)$ having x_0 as its ω-limit point, and L_α be a trajectory having x_0 as its α-limit point. These trajectories exist by the remark to Theorem 2. Consider an arbitrary arc C connecting points $x_\alpha \in L_\alpha$ and $x_\omega \in L_\omega$ inside $S(x_0, \eta)$. All points of the arc will belong to one of three classes: those having x_0 as their ω-limit point; those having x_0 as their α-limit point (these sets are closed); and hyperbolic trajectories relative to $S(x_0, \epsilon)$. Since trajectories of different classes pass through the ends of the arc and since each of these classes forms closed sets, the third class of hyperbolic trajectories must be non-empty.

The proven theorems lead to the following classification of simple saddle points: 1. **Elliptic saddle points.** A small neighborhood of such a point consists of elliptic and parabolic trajectories. 2. **Hyperbolic saddle points.** A small neighborhood of such a point consists of hyperbolic and parabolic trajectories. 3. **Elliptic-hyperbolic saddle points.** In every arbitrarily small neighborhood, trajectories of all three classes are present.

§ 1. CLASSIFICATION OF TOPOLOGICALLY STABLE POINTS

For linear equations with constant coefficients, stable singular points can only exist in even-dimensional spaces and are observed in the presence of purely imaginary roots of the characteristic equation. If the right-hand sides are homogeneous forms, stable points can likewise only exist in an even-dimensional space. Specifically, the following theorem by A. A. Shestakov [**?**] holds: Let the system be given where $i = 1, 2, \dots, n$. If n is an odd number and m is any number, or if n is any number and m is even, then the system under consideration has at least one integral straight line. If the requirement of homogeneity is dropped, the parity of the space' s dimension no longer plays a role. As early as 1937, S. K. Zaremba [**?**] constructed such an example in three-dimensional space. In this example, there is a sequence of nested tori, filled with everywhere dense windings, that contracts toward a singular point. A new example of this same phenomenon in three-dimensional space was constructed by A. D. Myshkis and L. E. Reizin [?]. Among stable points of general type, it is natural to distinguish the following:

Centers—points for which, for any $\epsilon > 0$, one can find a $\delta > 0$ such that every integral curve starting in the δ-neighborhood is a periodic or almost periodic solution contained within the ϵ-neighborhood. It is precisely these stable singular points that are observed for linear systems with constant coefficients.

Generalized centers—points in whose neighborhood there exists an integral whose level surfaces represent closed $(n-1)$-dimensional manifolds (the integrable case).

Center-foci—points in whose neighborhood there is a sequence of closed integral manifolds bounding regions that contain the singular point and contract toward it. A similar picture is presented by the system in S. K. Zaremba' s example [?].

SOME ANALYTICAL CRITERIA

Suppose that in a domain G, the uniqueness and existence of solutions to the Cauchy problem for the system $\frac{dx_i}{dt} = f_i(x_1, \dots, x_n)$ are guaranteed. Let the function $\Phi(x_1, \dots, x_n)$ be continuous and continuously differentiable everywhere in G, except perhaps at the singular points of the system, where Φ may be undefined. However, if Φ is defined at the singular points, we shall assume it to be continuous, continuously differentiable, and equal to zero there. Along with the function Φ, we consider the function $\dot{\Phi} = \sum \frac{\partial \Phi}{\partial x_i} f_i$, called the derivative of Φ by virtue of the system. The set of points where $\dot{\Phi} = 0$ will be called the neutral set.

If no other restrictions are placed on the pair of functions Φ and $\dot{\Phi}$, we shall say that Φ is a generalized Lyapunov function [?]. We shall call a generalized Lyapunov function "normal" if the neutral set does not contain any semi-trajectories of the system. Note that if Φ is normal, the set of zeros of this function also cannot contain entire semi-trajectories. If it is additionally assumed that $\dot{\Phi}$ is of constant sign in G, we call Φ a generalized Lyapunov-Krasovskii function; finally, if $\dot{\Phi}$ is sign-definite in G, then Φ is called a Lyapunov-Krasovskii function.

Lemma 1. Let a normal generalized Lyapunov-Krasovskii function be defined in domain G; then no ordinary point of the domain can be an ω-limit point of any semi-trajectory contained in G.

Suppose for definiteness that $\dot{\Phi} > 0$ in domain G, and assume for the sake of contradiction that p is an ordinary (non-singular) ω-limit point of the trajectory $f(r, t)$. Consider the trajectory $f(p, t)$, and let $\Phi(p) = \gamma$. On the trajectory, take a segment such that at point $p_1 = f(p, t_1)$, the function $\Phi(p_1) = \gamma_1 > \gamma$. Such a segment exists because $\dot{\Phi} > 0$, and even if $\dot{\Phi}(p) = 0$, we apply the condition of the theorem—no semi-trajectory lies on the neutral set. Let $\gamma_1 - \gamma = \delta > 0$.

Since Φ is a continuous function in G, one can find spheres $S(p, \epsilon)$ and $S(p_1, \epsilon)$ with centers at p and p_1 so small that in these neighborhoods:

$\Phi < \gamma + \frac{1}{3}\delta$ in $S(p, \epsilon)$ and $\Phi > \gamma + \frac{2}{3}\delta$ in $S(p_1, \epsilon)$.

Since p is also an ω-limit point for $f(r, t)$, there exists some T such that $f(r, T) \in S(p_1, \epsilon)$ and, consequently, $\Phi(f(r, T)) > \gamma + \frac{2}{3}\delta$. Therefore, this same inequality holds for all $t > T$, because $\dot{\Phi} > 0$ along the trajectory immersed in G. It follows from this that for $t > T$, the trajectory cannot enter $S(p, \epsilon)$, and thus p is not an ω-limit point, which contradicts the assumption.

Theorem. For a singular point to be simple, it is sufficient that there exists a normal generalized Lyapunov-Krasovskii function in some neighborhood U containing no other singular points.

Indeed, suppose such a Lyapunov-Krasovskii function exists in the neighborhood of point p, and let $f(r, t)$ be an arbitrary trajectory starting in U, and let p be an interior point of U. Based on the lemma, $f(r, t)$ cannot have ordinary limit points in G, and consequently, it either exits U or has p as a limit point; a similar argument applies to α-limit points.

The studies of P. N. Papush [**?**] and M. B. Kudaev [**?**] show that the simplicity of an unstable point can be established using other normal generalized Lyapunov functions. These functions themselves are positive definite, but their derivatives by virtue of the system change signs; the zero isocline $\dot{\Phi} = 0$ partitions the neighborhood into a finite number of regions.

Theorem. For a singular point to be a hyperbolic or parabolic unstable point, it is sufficient that there exists a normal generalized Lyapunov-Krasovskii function in some neighborhood, which is everywhere defined and continuous at the singular point.

By the definition of a Lyapunov-Krasovskii function, $\Phi(p) = 0$ at the singular point. Based on Theorem 4, it remains for us to show that in the case under consideration, elliptic trajectories cannot exist. Suppose for definiteness that $\dot{\Phi} > 0$. Assume for the sake of contradiction that $f(r, t)$ is a trajectory that enters p at both ends. Consider a point q on this trajectory where $\Phi(q) = \gamma \neq 0$; such a point exists, because otherwise the trajectory would lie on the zero level line and, consequently, on the neutral set. Consider two cases: in the first case, $\Phi(f(q, t)) > \gamma > 0$ for $t > 0$, and consequently, p cannot be an ω-limit point for $f(r, t)$. Let $\Phi(q) = \gamma < 0$; consider $f(r, t)$ for $t < 0$. Along this semi-trajectory, Φ decreases and, consequently, p cannot be an α-limit point for $f(r, t)$.

To distinguish between hyperbolic singular points and nodes, one must have some information about the structure of the level surfaces of the function Φ. For example, K. G. Danilin [**?**] showed that if Φ is sign-variable and the zero level line $\Phi = 0$ partitions the neighborhood of the point into a finite number of regions, and $\dot{\Phi} > 0$ everywhere except at the singular point, then p is a hyperbolic point.

Theorem. Let there exist a continuous Lyapunov-Krasovskii function in some neighborhood of an isolated singular point; then for the point to be parabolic, it is necessary and sufficient that Φ be of constant sign.

Suppose there exists a neighborhood in which $\Phi > 0$ and $\dot{\Phi} < 0$, except at

the point p, where $\dot{\Phi} = 0$. First of all, we note that $\Phi > 0$ at any point other than p. Indeed, if $\Phi(q) = 0$ for some $q \neq p$, then from the condition $\dot{\Phi} < 0$ in the neighborhood of p, it would follow that on an arc of the trajectory, Φ would have to be negative, which contradicts the condition of the theorem. Thus, in the neighborhood of point p, $\Phi > 0$ and $\dot{\Phi} < 0$; then, based on Lyapunov' s theorem, point p would be an asymptotically stable equilibrium position, and consequently, all curves exiting a sufficiently small neighborhood of p are parabolic.

We now prove the necessity of this condition. Specifically, we will show that under the conditions of the theorem, if we assume that Φ changes sign in an arbitrarily small neighborhood, then there exist hyperbolic curves relative to some fixed neighborhood. Indeed, consider a fixed neighborhood U in which $\dot{\Phi} < 0$, and denote by G^+ the set of points in this neighborhood where $\Phi > 0$, and G^- the set of points where $\Phi < 0$; then p is, by assumption, a limit point of both open sets. The boundaries of these sets consist of points where $\Phi = 0$ and points on the boundary of the neighborhood U. We shall call the boundary points belonging to $\Phi = 0$ the lateral surface of G^+ and G^-.

Let $S(p, \eta)$ be an arbitrarily small closed neighborhood of p, and let p^+ be a point in this neighborhood belonging to G^+. Consider $f(p^+, t)$. By Lemma 1, this trajectory must leave U after a finite time or have the singular point as an ω-limit point. However, the latter is impossible if it does not exit G^+, since at the singular point $\Phi(p) = 0$, while at point p^+ the function $\Phi(p^+) > 0$ and $\dot{\Phi} < 0$ in the domain G^+. Thus, $f(p^+, t)$ leaves G^+, but it cannot leave G^+ through the lateral surface because on the lateral surface $\Phi = 0$, while along the trajectory Φ decreases as t increases (or increases as t decreases). Considering a point p^- in $S(p, \eta)$ where $\Phi < 0$, we can similarly show that $f(p^-, t)$ leaves U. By connecting points p^+ and p^- with a continuous arc lying in $S(p, \eta)$ and reasoning as in Theorem 3, we prove that in any arbitrarily small neighborhood of point p, there are hyperbolic trajectories relative to any neighborhood U immersed with its boundary in G. This proves the theorem.

The example given in §1 shows that the set of parabolic curves can be reduced to just two curves. However, the following theorem holds.

Theorem. If in some neighborhood U there exists a sign-variable continuous Lyapunov-Krasovskii function such that the zero level (after excluding the singular point) decomposes into no fewer than two components, then the set of parabolic curves is no less than $(n-1)$-dimensional.

Let $\dot{\Phi} > 0$ everywhere in the neighborhood U except at the singular point, where $\dot{\Phi} = 0$. Let G^+ be a component (a set where $\Phi > 0$) to whose boundary the singular point p belongs. As already shown, whatever the semi-trajectory $f(p,t)$ may be, it goes to the boundary of U. Consider $f(p,t)$ for $t < 0$. We assume that at least two components of the zero level enter the boundary. We divide the boundary of G^+ into four sets: Γ_1 consists of one of the components of $\Phi = 0$, Γ_2 consists of all other components of

$\Phi = 0$ entering the boundary, and K consists of the points of the boundary entering the boundary of U. All these sets are non-empty. Consider $f^-(p,t)$, and let D be the set of those points p for which $f^-(p,t)$ has p as its α-limit point; this set D consists of parabolic points—we wish to estimate its dimension.

All other points are divided into three classes: A consists of those points p for which $f^-(p,t)$ leaves U through Γ_1; B consists of those points for which $f^-(p,t)$ leaves U through Γ_2 (these sets have no common points); finally, class C consists of hyperbolic curves for which $f^-(p,t)$ leaves U through K. We show that this set C cannot have the singular point as a limit point. Indeed, let $p_1, \dots, p_n, \dots$ be a sequence of points in C converging to p; then the semi-trajectory leaves U in arbitrary proximity to the points where $\Phi = 0$. Let their exit points converge to a point q belonging to the zero isocline. In this case, the trajectory $f(q,t)$ would be contained in the zero isocline, which contradicts the assumption $\dot{\Phi} > 0$ everywhere except at the singular point.

Exclude the set C and denote the remaining set by E. In the remaining set, there will be interior points, since G^+ was a domain and the boundary point p of this domain is not part of the boundary of U. The points of set E are now divided into two classes: A and B. Since A and B have no common points, the set D separates E (which has interior points) and is therefore at least $(n-1)$-dimensional. Theorem 6 obviously generalizes the aforementioned theorem by Danilin. Similar reasoning leads to the following lemma.

Lemma. Let G be a bounded domain whose boundary consists of two parts: Γ_1 belonging to the level surface of a continuous normal generalized Lyapunov-Krasovskii function $\Phi = 0$, where point p is an interior point relative to Γ_1, and part Γ_2. Then in this domain, there is a O-curve starting at p.

Suppose for definiteness that the function $\Phi > 0$. Consider on the lateral surface of the boundary Γ_1, i.e., on $\Phi = 0$, a sequence of points $q_1, \dots, q_n, \dots$ converging to the singular point p, and consider the positive semi-trajectories $f(q_n, t)$. Since $\dot{\Phi} > 0$ and $\Phi > 0$ inside the domain, the trajectory either remains on $\Phi = 0$ or enters the interior of G. But since Φ is a normal function, the trajectory does not have an arc on $\Phi = 0$ and, consequently, must enter G. Since $\dot{\Phi} > 0$ and $\Phi > 0$ in G, then as t increases, it cannot approach p (where $\Phi = 0$), cannot have ω-limit points in G (based on Lemma 1), and therefore exits G through Γ_2. Let $q'_1, \dots, q'_n, \dots$ be the exit points of $f(q_n, t)$ and let q^* be a limit point for the sequence q'_n. Then $f^-(q^*, t)$ will be the desired parabolic curve. Indeed, $f^-(q^*, t)$ remains indefinitely in G and therefore has an α-limit point in G; since Φ is continuous, this can only be p.

§ 3. EXAMPLES

Potential systems serve as the primary example of systems for which the study of integral curve behavior can be based on the construction of a Lyapunov-Krasovskii function.

We shall define the system $\dot{x}_i = P_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, n)$ as a potential system if the linear form $P_1 dx_1 + \cdots + P_n dx_n$ is the total differential of some function $\Phi(x_1, \ldots, x_n)$. In this case, if we take $\Phi(x_1, \ldots, x_n)$ as the generalized Lyapunov function, it automatically becomes a Lyapunov-Krasovskii function. Indeed, it follows that $\dot{\Phi} = \sum_{i=1}^{n} P_i \dot{x}_i = \sum_{i=1}^{n} P_i^2 \geq 0$. Consequently, equality holds only at those points where all $P_i = 0$, which corresponds exclusively to the singular points of the system. As is typical for Lyapunov' s second method, the same function remains a Lyapunov function for nearby systems. Let us introduce the following definition: the system $\dot{x}_i = Q_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ is said to be close to a potential system in a domain G if, within this domain, the following holds for all i:

$$|Q_i(x_1, x_2, \ldots, x_n) - P_i(x_1, x_2, \ldots, x_n)| < q|P_i(x_1, x_2, \ldots, x_n)|,$$

where $0 < q < 1$. The singular points of systems (1) and (2) obviously coincide.

Lemma 3. The function $V = -\Phi(x)$, which serves as a Lyapunov-Krasovskii function for the potential system $\dot{x}_i = P_i(x_1, \ldots, x_n)$, will also be a Lyapunov-Krasovskii function for the close system. Indeed, the derivative by virtue of system (2) takes the form $\dot{V} = -\sum \frac{\partial \Phi}{\partial x_i} Q_i = -\sum P_i Q_i$. Using the Cauchy inequality, we obtain:

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= 2 / , 2 (i - i) > o .

Regarding the singular points of a potential system, since the singular points of systems (1) and (2) coincide, the system vanishes only at the singular points of the system.

Example

Consider a system of the form:

$$\dot{x}_i = f_i(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n)$$

defined in some domain. For example, such a system could be represented as:

$$\dot{x}_i = x_i^k + \ldots \qquad (i = 1, 2, \ldots, s)$$
$$\dot{x}_i = \psi_i(x_1, \ldots, x_n) \quad (i = s + 1, \ldots, n)$$

where $\psi_i(x_1, \ldots, x_n)$ are convergent power series starting with terms of a certain order. The only singular point of this system is the origin.

Let us consider a sphere $S(0, \rho)$, where ρ is sufficiently small. By construction, the functions $f_i(x_1, \dots, x_n)$ are convergent series within a certain ball. Consequently, if the magnitude of the variables is small, specifically $|x| < \rho$, then $|f_i(x)|$ is also sufficiently small. Therefore, the system under consideration is close to a potential system.

d X ; = *? (i = l 9 2, s), dt

d X f = - x 1 (i = s + l , n). dt

The potential of this system is a function of $s + 2$ variables. Various cases may arise depending on the parity of the degrees. For example, if $s = n$ and all degrees are odd and positive, then $V > 0$ and $\dot{V} > 0$; consequently, the integral curves are parabolic.

If the degrees are odd and $s < n$, the manifold is homeomorphic to an $(n-1)$-dimensional plane, which partitions the space into two regions. In each of these regions, the trajectories will be O^+-curves, O^--curves, or hyperbolic curves, respectively. Example:

2. Consider the system

dx du dz = yz, —⌒- = zx, • • = xy. dt dt dt

This is a potential system, and its potential is a function

$y = xyz.$

Consequently, the zero surfaces will be the coordinate planes. Each octant will contain either O-curves or O^+-curves, while the remaining curves will be hyperbolic. One may consider a similar system: $\dot{x} = yz + \phi_1(x, y, z)$, $\dot{y} = zx + \phi_2(x, y, z)$, $\dot{z} = xy + \phi_3(x, y, z)$,

$\frac{dx}{dt} = yz + \phi_1(x, y, z), \quad \frac{dy}{dt} = zx + \phi_2(x, y, z), \quad \frac{dz}{dt} = xy + \phi_3(x, y, z),$

where ϕ_1, ϕ_2, ϕ_3 are series starting with terms of at least the second order with respect to x, y, z. Regarding this system, we can draw the same conclusions as for the potential system.

Example 3. Finally, these same methods can be applied to the classical case of studying the system $\dot{x} = Ax + \Phi(x, \dots)$. We assume that the matrix $A = \{a_{ij}\}$ is given in Jordan normal form, with sufficiently small ϵ on the superdiagonal, and $|\Phi(x_1, \dots, x_n)| \leq q(|x|)$, where $q(0) = 0$ and $q(\rho) \to 0$ as $\rho \to 0$. It is not difficult to establish that if the roots of the characteristic matrix have real parts of the same sign, then the Lyapunov-Krasovskii function in a sufficiently small neighborhood of the origin will be $V = \sum x_i^2$. If k roots have a positive real part and $n-k$ roots have a negative real part, the Lyapunov-Krasovskii function will be $V = \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2$. Consequently, in the first case, the singular point will be a node, and in the second case, it will be a hyperbolic saddle point.

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Note: Figure translations are in progress. See original paper for figures.