ON A NONLINEAR BOUNDARY-VALUE PROBLEM

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Abstract

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Let L be a simple smooth closed contour dividing the complex plane into the interior domain D^+ and the exterior domain D^-; suppose that the origin is enclosed by the contour L.

Denote by E^+ (E^-) the space of functions analytic in D^+ (D^-) and continuous in $\overline{D}^+$ ($\overline{D}^-$), with norm

$$\|\Phi^+(z)\| = \max_{z\in\overline{D}^+} |\Phi^+(z)|;$$

respectively

$$\|\Phi^-(z)\| = \max_{z\in\overline{D}^-} |\Phi^-(z)|.$$

Definition 1. We shall say (3) that a function $\omega(s)$ $(0 < s \le l)$ belongs to the class Φ^* if it satisfies the conditions:
1) $\omega(s)$ increases monotonically and is finite everywhere;
2) $\omega(s) \ne 0,\ \lim_{s\to 0}\omega(s) = 0$;
3) there exists a constant $\tilde{c} > 1$ such that

$$1 < \lim_{s\to 0}\frac{\omega(cs)}{\omega(s)} \le \overline{\lim}_{s\to 0}\frac{\omega(cs)}{\omega(s)} < \tilde{c}.$$

Definition 2. $u(t) \in H_k(\omega)$, $t \in L$, if $|u(t)| \le k$, $|u(t_1) - u(t_2)| \le k\omega(|t_1 - t_2|)$, $t_1, t_2 \in L$, where $k = \text{const}$, $\omega(s) \in \Phi^*$.

Statement of the problem. It is required to find a function $\Phi^+(z) \in E^+$ and $\Phi^-(z) \in E^-$ satisfying the boundary condition

$$[\Phi^+(t)]^n + F\left(t, \int_L \frac{f[\tau, \Phi^+(\tau)]}{\tau - t}\,d\tau\right) = G(t)\Phi^-(t), \qquad (*)$$

where $n \geq 2$ is an integer, $G(t)$ is a given function of the class $H_k(\omega)$ on L and is nowhere zero on L; $f(t,u)$ is a function defined for $t \in L$ and $u = \Phi^+(z) \in E^+$, satisfying the condition:

$$|f(t_1,u_1) - f(t_2,u_2)| \leq A[\omega(|t_1 - t_2|) + (u_1 - u_2)],$$

$$\omega(s) \in \Phi^*, \quad 0 < s \leq l, \quad A = \text{const} > 0, \tag{1}$$

$F(t,v)$ is a function defined for $t \in L$ and $v = \int_L \frac{f(\tau,u)}{\tau - t}\, d\tau,\ u = \Phi^+(z) \in E^+$, satisfying the conditions:

$$|F(t,v)| \leq B_0(1 + |v|^{n-\varepsilon}), \quad 0 < \varepsilon < n; \tag{2}$$

$$|F(t_1,v_1) - F(t_2,v_2)| \leq B_0[(1+\tilde{v}^{n-\varepsilon})\omega(|t_1 - t_2|) + (1+\tilde{v}^{n-1-\varepsilon})|v_1 - v_2|], \quad \tilde{v} = \max(|v_1|,|v_2|). \tag{3}$$

If one takes (1) into account, then it can be shown that

$$|F(t,v)| \leq B(1 + |u|^{n-\varepsilon}), \tag{4}$$

$$|F(t_1,v_1) - F(t_2,v_2)| \leq B\big[(1+\tilde{u}^{n-\varepsilon})\omega(|t_1 - t_2|) + (1+\tilde{u}^{n-1-\varepsilon})|su_1 - su_2|\big], \tag{5}$$

where $B = \text{const} > 0$, $\tilde{u} = \max(|u_1|,|u_2|)$,

$$Su = \int_L \frac{f(\tau,u)}{\tau - t}\, d\tau.$$

The operator defined in E^+ by the function

$$F\left(t, \int_L \frac{f(\tau,\Phi^+)}{\tau - t}\, dt\right)$$

will be denoted simply by $FS\Phi^+$. If we make the substitution

$$\Phi_1^+(z) = [\Phi^+(z)]^n, \qquad \Phi_1^-(z) = \Phi^-(z)$$

and use the theory of the linear Riemann boundary-value problem $(1,2,4)$, then we shall have

$$\Phi^+(z) = e^{\Gamma^+(z)/n}\,[P_\chi(z) - H^+FS\Phi^+]^{1/n},$$

$$\Phi^-(z) = e^{\Gamma^-(z)}[P_\chi(z) - H^-FS\Phi^+], \tag{6}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln[t^{-\chi}G(t)]}{t-z}\,dt, \qquad H\varphi = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}e^{-\Gamma^+(t)}\,dt, \qquad z \in D^+ \cup D^-.$$

Suppose that $\Phi^+(z)$ has no branch points in D^+ and that $\chi = \operatorname{ind} G(t) \geq 0$. Then problem (6) actually splits into n problems corresponding to the different branches of the radical, and the equivalence of problems (*) and (6) is obvious; therefore it is enough to prove the solvability of problem (6).

Put $P_\chi(z) \equiv c$ and consider in the space E^+ the operator

$$W\Phi^+ = e^{\Gamma^+(z)/n}[c - H^+FS\Phi^+]^{1/n},$$

where a certain branch is fixed.

Denote by S_R the closed sphere in E^+ of radius R: $\|\Phi^+(z)\| \leq R$; $S_R(N)$ is the set of functions from S_R whose boundary values belong to the class $H_N(\omega)$.

Lemma 1. If the function $f(t,u)$ satisfies condition (1), then the function

$$v(t) = \int_L \frac{f[\tau, u(\tau)]}{\tau - t}\,d\tau$$

for $u(t) \in H_N(\omega)$ satisfies the condition

$$|v(t_1) - v(t_2)| \leq DA(1+N)\omega(|t_1 - t_2|),$$

where $D = \text{const}$ does not depend on A and N.

Lemma 2. If the function $f(t,u)$ satisfies condition (1), and the function $F(t,v)$ satisfies conditions (2), (3), or, equivalently, conditions (4), (5), then the operator

$$H^*u = \frac{1}{2\pi i}\int_\Gamma \frac{F\left[\tau, \displaystyle\int_\Gamma \frac{f(\tau_1, u)}{\tau_1 - \tau}\,d\tau_1\right]}{\tau - t}e^{-\Gamma^+(\tau)}\,d\tau$$

is continuous on the set $S_R(N)$.

Proof. Consider the operators

$$Su = \int_L \frac{f[\tau, u(\tau)]}{\tau - t}\, d\tau, \tag{7}$$

$$S^* v \int_L \frac{F[\tau, v(\tau)]}{\tau - t} e^{-\Gamma^+(\tau)}\, d\tau. \tag{8}$$

To prove the validity of the lemma, it is evidently sufficient to prove the continuity of the operators (7) and (8). Let us prove the continuity of the operator Su.

If (1) is taken into account and $u(t) \in H_N(\omega)$, then it is easy to see that there exists a constant $M^* > 0$ such that $|f[\tau, u(\tau)]| \leq M^*$, $\tau \in L$; $|f[\tau_1, u(\tau_1)] - f[\tau_2, u(\tau_2)]| \leq M^* \omega(|\tau_2 - \tau_1|)$, $\tau_2, \tau_1 \in L$. Then there exists a constant $A^* > 0$, independent of M^*, such that

$$Su = \left| \int_L \frac{f[\tau, u(\tau)]}{\tau - t}\, d\tau \right| \leq M^* A^*. \tag{9}$$

Let

$$\max_{t \in L} |u_n(t) - u(t)| = \varepsilon.$$

We have

$$Su_n - Su = \int_L \frac{f[\tau, u_n(\tau)] - f[\tau, u(\tau)]}{\tau - t}\, d\tau,$$

$$|f[\tau, u_n(\tau)] - f[\tau, u(\tau)]| =$$

$$= |f[\tau, u_n(\tau)] - f[\tau, u(\tau)]|^{\mu} |f[\tau, u_n(\tau)] - f[\tau, u(\tau)]|^{1-\mu} \leq$$

$$\leq A_1^{\mu} \varepsilon^{\mu} (2M^*)^{1-\mu} \leq \widetilde{M} \varepsilon^{\mu} \qquad (0 < \mu < 1);$$

$$|f[\tau_1, u_n(\tau_1)] - f[\tau_1, u(\tau_1)] - f[\tau_2, u_n(\tau_2)] + f[\tau_2, u(\tau_2)]| \leq$$

$$\leq 2\mu A_1^{\mu} \varepsilon^{\mu} [2M^* \omega(|\tau_2 - \tau_1|)]^{1-\mu} \leq \widetilde{M} \varepsilon^{\mu} [\omega(|\tau_2 - \tau_1|)]^{1-\mu},$$

where $\widetilde{M} = 2A_1^{\mu} (M^*)^{1-\mu}$, $\tau_1, \tau_2 \in L$.

Then, according to (9),

$$|Su_n - Su| \leq \varepsilon^{\mu} \widetilde{M} \widetilde{A},$$

where $\widetilde{A} = \text{const}$ does not depend on $\widetilde{M} \varepsilon^{\mu}$.

The continuity of the operator $S^* v$ is proved analogously.

Lemma 3. Let the function $f(t, u)$ satisfy condition (1), $f(t, v)$ conditions (4), (5), and let $u = \Phi^+(z) \in S_R(N)$. Then the function

$$v(t) = H^+ F S u \equiv \frac{1}{2\pi i} \int_L \frac{F\left\{\tau, \int_L \frac{f[\tau_1, u(\tau_1)]}{\tau_1 - \tau}\, d\tau_1\right\}}{\tau - t} e^{-\Gamma^+(\tau)}\, d\tau$$

belongs to the class $H_{\widetilde{K}}(\omega)$, where $\widetilde{K} = 2\widetilde{B}BR^{n-1-\varepsilon}\{[R+DA(1+N)]M+rkR\}$, $M = \max(\|e^{\Gamma^+(z)}\|, e^{-\Gamma^-(z)}, 1)$, and $\widetilde{B}, r$ are constants independent of R and N.

Lemma 4. If $f(t,u)$ satisfies condition (1), $F(t,v)$ conditions (4) and (5), $R = P(|c|)^{1/n}$, $N = q(|c|)^{1/n}$, $|c|^{\varepsilon/n} > \max(\tilde{k}_1, \tilde{k}_2)$, where

$$\tilde{k}_1 = 6\widetilde{B}p^{n-1}M\left\{p + [DA(p+q)+p]\frac{1}{\pi m}\int_0^{l/2}\frac{\omega(s)}{s}\,ds\right\},$$

$$\tilde{k}_2 = \frac{3}{2}(1+2\widetilde{B})Bp^{n-1}\{[p+DA(p+q)]M+rkp\},$$

$p,\ q$ are numbers ≥ 1, $m = \text{const}$, then $WS_R(N) \subset S_{R_1}(N_1)$, where $R_1 = [2M|c|]^{1/n}$, $N_1 = (rk+1)\times[2M|c|]^{1/n}$.

Lemma 5. If all the conditions of Lemma 4 are satisfied, then the operator $W\Phi^+$ is continuous on $S_R(N)$.

By virtue of the lemmas indicated above, the following is established.

Theorem. If $f(t,u)$ satisfies condition (1), and $F(t,v)$ satisfies conditions (2) and (3), or, equivalently, conditions (4) and (5), and $\chi = \operatorname{ind} G(t) \geqslant 0$, then problem $(*)$ is solvable.

Remark 1. The assumption $\chi \geqslant 0$ in the above arguments is essential, since the solvability of problem $(*)$ is to a considerable extent ensured by the presence of the parameter c.

Remark 2. A problem of the form

$$\Phi^+(t) = G(t)[\Phi^-(t)]^n + F\left(t, \int_L \frac{f[\tau,\Phi^-(\tau)]}{\tau - t}\,d\tau\right) \qquad (*')$$

by means of the change of variable $z = 1/w$ ($\tau = 1/t$) is transformed into the problem

$$[\Phi_1^+(\tau)]^n + F_1\left(\tau, \int_L \frac{f_1[\tau_1,\Phi_1^+(\tau_1)]}{\tau_1 - \tau}\,d\tau_1\right) = G_1(\tau)\Phi_1^-(\tau),$$

where

$$f_1[\tau_1,\Phi_1^+(\tau_1)] = f\left[\frac{1}{\tau_1},\Phi_1^+(\tau_1)\right]\frac{\tau}{\tau_1}, \qquad F_1(\tau, S\Phi_1^+) = F\left(\frac{1}{\tau}, S\Phi_1^+\right)\Big/ G\left(\frac{1}{\tau}\right).$$

Further, note that

$$\operatorname{Ind} G_1(\tau) = \operatorname{Ind}\frac{1}{G(1/\tau)} = \operatorname{Ind} G(t) = \chi.$$

It is not hard to see that if $f(t,u)$ satisfies condition (1), and $F(t,v)$ satisfies conditions (2) and (3), or, equivalently, (4) and (5), then $f_1(\tau,u)$ and $F_1(\tau,v)$, respectively, satisfy analogous conditions.

Consequently, a problem of the form $(*')$ is also solvable.

Remark 3. One can prove the solvability also of the more general problem

$$[\Phi^+(t)]^n + F\left\{t, \Phi^+(t), \int_L \frac{f[t,\tau,\Phi^+(\tau)]}{\tau - t}\, d\tau\right\} = G(t)\Phi^-(t), \qquad (**)$$

where the function $f(t,\tau,u)$ is defined for $t,\tau \in L$, $u = \Phi^+(z) \in E^+$, and

$$|f(t_1,\tau_1,u_1) - f(t_2,\tau_2,u_2)| \le A\{\psi(|t_1-t_2|) + \omega(|\tau_1-\tau_2|) + |u_1-u_2|\},$$

$$\omega(s),\ \psi(s) \in \Phi^*, \qquad \psi(s)|\ln s| \le F_0\omega(s), \qquad 0 < s \le l, \qquad F_0 = \text{const} > 0;$$

$F(t,u,v)$ is defined for $t \in L$, $u = \Phi^+(z) \in E^+$,

$$v = \int_L \frac{f(t,\tau,u)}{\tau - t}\, d\tau$$

and satisfies the conditions

$$|F(t,u,v)| \le B_1(1 + |u|^{n-\varepsilon} + |v|^{n-\varepsilon}), \qquad 0 < \varepsilon < n,$$

$$|F(t_1,u_1,v_1) - F(t_2,u_2,v_2)| \le B_1\{(1-\tilde{v}^{n-\varepsilon})\omega(|t_1-t_2|) + (1+\tilde{v}^{n-1-\varepsilon})[|u_1-u_2| + |v_1-v_2|]\},$$

$$\tilde{v} = \max(|u_i|, |v_i|), \qquad i = 1,2.$$

A problem of the form $(**)$, when the function F does not depend on the third expression, in the Hölder class $\tilde{H}_\mu(\delta)$, was considered in paper (6).

In conclusion we report that, using paper (5), it has been proved that a solution of problem $(*)$ can be obtained by the method of successive approximations.

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Note: Figure translations are in progress. See original paper for figures.