QUASI-RANDOM OSCILLATIONS OF A ONE-DIMENSIONAL OSCILLATOR

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Abstract

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MATHEMATICS

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QUASI-RANDOM OSCILLATIONS OF A ONE-DIMENSIONAL OSCILLATOR

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1. It is known that the behavior of geodesics on a surface of negative curvature can be studied by means of the methods of "symbolic dynamics" (see, for example, ([1]), Ch. VIII, § 11). S. Smale ([2]) showed that these methods can be of substantial use also in general problems of the qualitative theory of differential equations on manifolds of dimension $\geqslant 3$. Although the ideas of the construction used by Smale are of a very general character, their realization is nevertheless closely connected with mappings of a special kind; therefore verification of the presence of a "Smale situation" in concrete cases requires considerable effort and, probably, is not always possible. In particular, the author was not able directly to apply the assertions of paper ([2]) to the study of equation (1) (see below), although, undoubtedly, the ideas of the proof of Theorem 2 are quite close to the idea of Smale' s construction.

The basis of the methods of "symbolic dynamics" is the assignment, to trajectories of a dynamical system, of infinite words over some alphabet. For this purpose, on a cross-section surface in phase space one chooses a certain partition, whose elements are declared to be letters. Marking the elements into which the trajectory falls as it successively returns to the cross-section surface, we obtain the word corresponding to the given trajectory. It is clear that the basic mapping of the cross-section surface—the sequence function—then corresponds to a shift by one letter to the left. This scheme has the following physical interpretation. The letters, i.e. the elements of the partition, correspond to the outcomes of a certain experiment, a "macro-measurement" ; the word corresponds to the sequence of results of repeated measurements in a system evolving with time. The greatest interest, of course, is presented here by the case when the partition is coarse, i.e. can be produced by means of a smooth function on the phase space, preferably, naturally connected with the given system.

A system may be called quasi-random if the stock of admissible words, i.e. words to which some trajectory corresponds, is sufficiently large, so that, observing the course of the process for an arbitrarily long time, we still cannot predict its subsequent behavior. By assigning a suitable measure on the phase space, we can model a random process in a quasi-random system. If every infinite word is admissible, then the probabilities of occurrence of letters at different places can be made independent (in the probabilistic sense) and invariant under a shift of the word. This gives a realization of the "Bernoulli scheme" (see below the remark after Corollary 4).

The theorems formulated below show that quasi-randomness occurs for solutions of differential equations of second order that describe the one-dimensional motion of a particle in a potential well periodically varying with time. The conditions imposed on the force field are such that the particle cannot escape to infinity and its motion is oscillatory in character, but, generally speaking, will not be periodic. If the time of return of the particle to the center of the well grows sufficiently rapidly with increasing amplitude of oscillation, then the phase in which

the system will find itself at the moment of return changes very strongly under a small change of the initial conditions. Therefore the intervals between successive returns have a quasi-random character.

2. Let us consider the differential equation

$$\ddot{x} = -Q(t, x), \tag{1}$$

assuming the following conditions to be satisfied:

1°. $Q(t,x), Q'_x, Q'_t$ and Q''_{tt} are continuous for all t and x.

2°. Q is odd in x and has period 2π in t.

$$3^\circ.\quad Q(t,x) > 0 \quad \text{for } x > 0; \qquad \int_0^\infty \int_0^{2\pi} Q(t,x)\,dt\,dx = \infty.$$

$$4^\circ.\quad |Q''_{tt}(t,x)| \leqslant \psi(x); \qquad \int_0^\infty x\psi(x)\,dx < \infty.$$

$$5^\circ.\quad x\psi(x)/Q(t,x)^2 \to 0 \quad \text{as } x \to +\infty \text{ uniformly in } t.$$

6°. $Q'_x \leqslant 0$ for $x \geqslant x^* > 0$.

Conditions 2° and 3° correspond to the presence of a potential well from which the particle cannot escape to infinity. Conditions 4° and 5° are of a purely technical character and are connected with the details of the proof. Condition 6° is an analogue of negative curvature in the geodesic problem. Since we require only that it be satisfied for sufficiently large x, the behavior of the solutions in our problem is analogous to the behavior of geodesics on a surface whose curvature may be partly positive.

Denote by $x(t; v, \tau)$ the solution of equation (1) determined by the initial conditions

$$x(\tau; v, \tau) = 0, \qquad \dot{x}(\tau; v, \tau) = v.$$

It follows immediately from 2° that

$$x(t + 2\pi; v, \tau) \equiv x(t; v, \tau + 2\pi), \qquad x(t; -v, \tau) \equiv -x(t; v, \tau).$$

Therefore, in studying the properties of the solutions it is sufficient to restrict ourselves to the case $v \geqslant 0$ and $\tau(\mathrm{mod} 2\pi)$. We shall regard (v, τ) as polar coordinates of an auxiliary plane Φ; in particular, the points $(0, \tau)$ are identified with one point O (it corresponds to the trivial solution $x \equiv 0$, i.e. a particle at rest at the center of the well). Define the successor function S: $(v, \tau) \to (v', \tau')$, where τ' is the zero of the solution $x(t; v, \tau)$ nearest to τ on the right, and $v' = -\dot{x}(\tau'; v, \tau)$. In addition, by definition, $S(O) = O$. It is easy to prove that S is a homeomorphism of Φ onto itself and preserves area.

Between neighboring zeros τ and τ' the solution $x(t; v, \tau)$ attains a maximum $X^+(v, \tau)$. Similarly, between τ and τ''—the zero nearest to τ on the left—it attains a minimum, the absolute value of which we denote by $X^-(v, \tau)$. Let

$$Q_0(x) = \frac{1}{2\pi} \int_0^{2\pi} Q(t, x)\, dt \quad \text{and} \quad h^\pm(v, \tau) = \int_0^{X^\pm(v,\tau)} Q_0(x)\, dx.$$

In the stationary case $h^+(v, \tau) \equiv h^-(v, \tau) \equiv v^2/2$, and the common value of these functions is the energy of the oscillation. It can be shown that also in the nonstationary case the level lines $h^+ = C$ and $h^- = C$ are smooth Jordan curves for $C > 0$, containing the point O inside them, and from the definition it follows immediately that the successor function carries one of these lines into the other. The fixed points of the successor function therefore lie at their intersections.

Theorem 1. To each simple root τ_0 of the equation

$$\int_0^\infty x Q_t'(t, x)\, dx = 0 \tag{2}$$

there corresponds a smooth branch of the curve $h^+(\nu, \tau) = h^-(\nu, \tau)$, having the ray $\tau = \tau_0$ as its asymptote. On this branch there exists a sequence of points (ν_n, τ_n) such that, for all $n \geq n(\tau_0)$, the solution $x(t; \nu_n, \tau_n)$ has period $4\pi n$, antiperiod $2\pi n$; on the interval $(\tau_n, \tau_n + 2\pi n)$ this solution is > 0.

3. Let $\tau_1, \tau_2, \ldots, \tau_k$ be simple roots of equation (2). According to Theorem 1, to any pair (m, n), where $1 \leq m \leq k$ and n is sufficiently large, there corresponds a point $P_{m,n} = (\nu_{m,n}, \tau_{m,n})$, generating a periodic solution.

To arbitrary N and an arbitrary function $\varphi(n)$ we associate the graph $G(N, \varphi)$, whose vertices are the points $P_{m,n}$, $1 \leq m \leq k$, $n \geq N$, and whose edges are all possible pairs of vertices $[P_{m,n}, P_{m',n'}]$ such that $|n'-n| \le \varphi(n)$. An arbitrary sequence of vertices $\{P^{(l)}\}$, infinite in both directions, forms a path in the graph $G(N,\varphi)$ if every pair $[P^{(l)}, P^{(l+1)}]$ is an edge. Let also

$$U(P_0,\varepsilon) = \{(\nu,\tau), |\nu-\nu_0| < \varepsilon,\ |\tau-\tau_0| < \varepsilon\},$$

where $P_0 = (\nu_0, \tau_0)$.

Theorem 2. If

$$\lim_{x\to+\infty} Q_0(x) \int_0^x Q_0(y)\,dy = 0, \tag{3}$$

then there exists a function $\varphi(n) \uparrow \infty$ and, for every $\varepsilon > 0$, an $N(\varepsilon,\varphi)$ such that to every path

$$\pi = \{P^{(l)}\}, \qquad -\infty < l < +\infty,$$

in the graph $G(N,\varphi)$ there corresponds a point P such that $S^l(P) \in U(P^{(l)},\varepsilon)$. If the path π is periodic, then the point P corresponds to a periodic solution of equation (1).

In what follows we shall assume that, in addition to conditions 1°-6°, the following condition is also satisfied:

7°. Equation (2) has at least one simple root and (3) is satisfied.

Corollary 1. Equation (1) has a continuum of solutions asymptotic to periodic ones, and a countable number of doubly asymptotic solutions (i.e., asymptotic as $t \to +\infty$ to one periodic solution, and as $t \to -\infty$ to another).

These solutions correspond to paths $\{P^{(l)}\}$ in the graph that are periodic for $l \ge l_0$ ($l \le l_1$), or simultaneously for $l \ge l_0$ and $l \le l_1$.

Corollary 2. If $P_n(\nu_n, \tau_n)$ are the same as in Theorem 1, then, for sufficiently large n, in any neighborhood $U(P_n,\varepsilon)$ one can find a point (ν,τ) such that

$$\sup_{t<\tau} |x(t;\nu,\tau)| = \infty.$$

For this it is enough to take a path such that $P^{(k)} = P_n$, $k < 0$, and $P^{(k)} = P_{n+k}$, $k \ge 0$. The corresponding solution as $t \to -\infty$ is asymptotic to $x(t;\nu_n,\tau_n)$, while as $t \to +\infty$ the amplitude of its oscillations grows without bound.

Corollary 3. For any $\varepsilon > 0$ and $A > 0$ there exists an N such that, for any sequence $\{n_k\}$ with $n_k \ge N$, $|n_k - n_{k+1}| \le A$, there exists a solution $x(t)$ whose consecutive zeros $\{t_k\}$ satisfy the condition

$$|(t_{k+1}-t_k)/2\pi - n_k| < \varepsilon.$$

To construct such a solution one must take the path $\pi = \{P_{n_k}\}$, where the poin-

the $P_{n_k} = (v_{n_k}, \tau_{n_k})$, according to Theorem 1, generates a solution with antiperiod $2\pi n_k$, while the τ_{n_k} tend to the simple root of equation (2).

4. Let $X = \{1, 2, \dots, p\}$ be a discrete space of p points, and $\mathfrak{X}^p = \{x;\ x = [x_n],\ -\infty < n < +\infty,\ x_n \in X\}$ the Tikhonov product of a countable number of copies of the space X. The shift transformation $B[x_n] = [x_{n+1}]$ is a homeomorphism $B : \mathfrak{X}^p \to \mathfrak{X}^p$.

Corollary 4. *For every p there exists a homeomorphism $\varphi : \mathfrak{X}^p \to \Phi$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{X}^p & \xrightarrow{B} & \mathfrak{X}^p \\ \varphi \downarrow & & \downarrow \varphi \\ \Phi & \xrightarrow{S} & \Phi \end{array}$$

is commutative.

Endowing each X with one and the same measure P such that $\mathrm{P}\{x_k\} = p_k$, and taking the Cartesian product of these measures, we obtain a Bernoulli scheme. From Corollary 4 it follows that in Φ there is a subset (compact set) invariant with respect to S and isomorphic to the space of realizations of the Bernoulli scheme.

The preceding assertion can be somewhat generalized, namely:

Theorem 3. *For any Markov process with a countable number of states $\{1, 2, \dots, n, \dots\}$, whose transition probabilities for some A satisfy the condition $p_{mn} = 0,\ |m - n| > A$, there exist a function $f(P) \in C^\infty(\Phi)$ and a measure P such that the sequence of functions $f_n(P) = f(S^n(P))$ on the space with measure $\{\Phi, \mathrm{P}\}$ is isomorphic to the given Markov process.*

The function $f(P) = f(v, \tau)$ may be taken arbitrarily close to the function $(\tau' - \tau)/2\pi - N$, differing only by a constant N from the number of periods between successive returns of the particle to the point $x = 0$.

5. Theorems 1-3 are applicable, for example, to the case $Q(t, x) = Q_0(x) + Q_1(t, x)$, where Q_0 satisfies $2^\circ, 3^\circ, 6^\circ$ and (3) (for the latter it is sufficient to have the power asymptotic at infinity: $Q_0(x) \sim Cx^\alpha,\ -1 \le \alpha < -1/2$), and Q_1 is localized in an arbitrarily narrow strip $x_0 \le x \le x_0 + \varepsilon$. In particular, arbitrarily small periodic perturbations near the center of a stationary potential well give rise to resonance in long-period oscillations, imparting to them a quasirandom character.

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Note: Figure translations are in progress. See original paper for figures.