CHRONOMETRIC INVARIANCE AND THE PROBLEM OF GRAVITATIONAL ENERGY

PHYSICS

1967

SovietRxiv

Abstract

Full Text

UDC 530.12:531.51

PHYSICS

N. V. MITSKEVICH, JOSÉ D. MUJICA

CHRONOMETRIC INVARIANCE AND THE PROBLEM OF GRAVITATIONAL ENERGY

(Presented by Academician V. A. Fock, 24 XII 1966)

Relativistic laws manifestly possess four-dimensional space-time symmetry; however, the distinguished role of time is a fundamental feature of the physical world, and for this reason, in particular, the concept of a reference system plays an important role. Moreover, in a curved world one cannot say anything about the system under consideration from coordinates alone; for this it is necessary to specify the field of the metric tensor. In view of this freedom of the formal choice of coordinates, we shall follow the approach of A. L. Zel' manov ([1]) (see also ([2,3])) and shall assume that all coordinate systems related to one another by transformations

$$x'^0 = x'^0(x^0, x^1, x^2, x^3), \tag{1}$$

$$x'^i = x'^i(x^1, x^2, x^3), \tag{2}$$

where $\partial x'^i / \partial x^0 = \partial x^i / \partial x'^0 = 0$, i.e., mutually at rest, belong to one and the same reference system. It is obvious that transformations (1) and (2), without taking us outside the original reference system, cannot lead to changes in the observed picture of the world. Therefore we shall require that all observable quantities be invariant with respect to transformations (1)—chronometrically invariant; with respect to transformations (2) they may be three-dimensional tensors of various ranks (ch. tensors), and also three-objects (ch. objects).

Einstein' s definition of gravitational energy ([4]) encountered well-known difficulties ([5–7]), since the unconditionally necessary invariance of energy with respect to transformations (2) was not satisfied. It was precisely such invariance that Møller ([7]) adopted as the basic requirement; however, he rejected transformations of the coordinate time, taking $x'^0 = x^0$. An investigation of the properties of a number of expressions for the energy density in general relativity, introduced in works ([4,8–10,7]), showed ([11]) that all these expressions fail to satisfy Zel' manov' s requirement of chronometric invariance.

Energy is one of the observable quantities. It is therefore natural to consider precisely it (and not its density) as a ch. scalar. Integration of the ch. scalar quantity of the energy element

$$dE = w^{\sigma} dS_{\sigma} \tag{3}$$

over a three-volume (a spacelike hypersurface, whose element is dS_{σ}) will be a completely correct operation, since the transfer of a three-scalar in the three-dimensional world singled out by the hypersurface is unambiguous. This removes one of the principal difficulties in formulating the concept of integral energy in a Riemannian space, caused by the absence of distant parallelism in the transport of tensors. We shall show here that this program is feasible even in the most general case.

Starting from the formulation of Noether' s theorem ([10]), we shall assume that the invariant Lagrangian $L = \mathfrak{L}/\sqrt{-g}$ depends on the field potentials, their first and second derivatives. Taking the Lagrangian of the complete system of fields, so that the equations of all fields are assumed to be satisfied, we obtain weak conservation laws. Consider infinitesimal, triply differentiable coordinate transformations $x'^{\mu} = x^{\mu} + \xi^{\mu}$. The potentials are then transformed according to the law $\delta A_B = a^{\tau}_{B\sigma} \xi^{\sigma}{}_{,\tau}$ (a comma denotes partial differentiation). The infinitesimal vector ξ^{μ} can be decomposed into a ch.-scalar $\tau = \xi_0/\sqrt{g_{00}}$ and a ch.-vector $\lambda^i = \xi^i$. It is obvious that only the zero ch.-vector can be constant in the 3-covariant sense, so that here we shall put $\lambda^i = 0$ and therefore shall not now consider the problem of conservation of 3-momentum. As for energy, in the spirit of Noether' s theorem we adopt the ch.-scalar condition $\tau = \text{const}$. Substituting this condition into Noether' s relation ([10]),

$$\left\{ t^{\alpha}_{\sigma} \xi^{\sigma} - \mathfrak{M}^{\alpha\tau}_{\sigma} \xi^{\sigma}{}_{,\tau} - \mathfrak{N}^{\alpha\tau\beta}_{\sigma} \xi^{\sigma}{}_{,\tau,\beta} \right\}_{,\alpha} = 0, \tag{4}$$

where the expression in braces is the true density of a contravariant 4-vector; the canonical energy-momentum quasitensor is

$$t^{\alpha}_{\sigma} = \frac{\delta \mathfrak{L}}{\delta A_{B,\alpha}} \cdot A_{B,\sigma} + \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha,\beta}} \cdot A_{B,\sigma,\beta} - \mathfrak{L} \delta^{\alpha}_{\sigma};$$

the density of generalized spin is

$$\mathfrak{M}^{\alpha\tau}_{\sigma} = \frac{\delta \mathfrak{L}}{\delta A_{B,\alpha}} \cdot a^{\tau}_{B\sigma} - \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha,\tau}} \cdot A_{B,\sigma} + \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha,\beta}} \cdot a^{\tau}_{B\sigma,\beta},$$

and the density of biskew is

$$\mathfrak{N}^{\alpha\tau\beta}_{\sigma} = \frac{1}{2} \left(\frac{\partial \mathfrak{L}}{\partial A_{B,\alpha,\beta}} \cdot a^{\tau}_{B\sigma} + \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha,\tau}} \cdot a^{\beta}_{B\sigma} \right).$$

We then obtain the conservation law for a one-index quantity (the energy density)

$$w^\sigma = \frac{1}{\sqrt{g_{00}}} \cdot t_0^\sigma - \left(\frac{1}{\sqrt{g_{00}}}\right)_{,\alpha} \cdot \mathfrak{M}_0^{\sigma\alpha} - \left(\frac{1}{\sqrt{g_{00}}}\right)_{,\alpha,\beta} \cdot \mathfrak{N}_0^{\sigma\alpha\beta}; \tag{5}$$

$$w^\sigma{}_{,\sigma} = 0. \tag{6}$$

This quantity behaves under transformations (1) and (2) as a ch.-scalar multiplied by the density of a contravariant 4-vector. This assertion follows from the properties of expression (4) indicated above. Moreover, it is not difficult to verify it also from the exact transformation properties of the dynamical quantities under general coordinate transformations ([12]):

$$t'^{\alpha}_{\sigma} = |J|^{-1} \cdot \frac{\partial x'^\alpha}{\partial x^\beta} \cdot \left[t_\tau^\beta - \mathfrak{M}_\tau^{\beta\varepsilon} \frac{\partial}{\partial x^\varepsilon} - \mathfrak{N}_\tau^{\beta\varepsilon\omega} \frac{\partial^2}{\partial x^\varepsilon \partial x^\omega} \right] \frac{\partial x^\tau}{\partial x'^\sigma};$$

$$\mathfrak{M}'^{\alpha\lambda}_{\sigma} = |J|^{-1} \cdot \frac{\partial x'^\alpha}{\partial x^\beta} \cdot \left[\frac{\partial x'^\lambda}{\partial x^\nu} \left(\mathfrak{M}_\tau^{\beta\nu} + 2\mathfrak{N}_\tau^{\beta\mu\nu} \frac{\partial}{\partial x^\mu} \right) \frac{\partial x^\tau}{\partial x'^\sigma} + \frac{\partial x^\tau}{\partial x'^\sigma} \frac{\partial^2 x'^\lambda}{\partial x^\mu \partial x^\nu} \mathfrak{N}_\tau^{\beta\mu\nu} \right];$$

$\mathfrak{N}_\sigma^{\alpha\lambda\beta}$ is the true density of a 4-tensor; together these quantities form a 4-object. Thus, the energy differential (the energy of an element of 3-volume) (3) is a ch.-scalar (strictly speaking, an axial one, as it must be), and from the differential conservation law (6) there follows the integral law

$$E = \int_\Sigma w^\alpha dS_\alpha = \text{const} \tag{7}$$

(if a closed system is considered). From Noether' s relations ([10]) it follows that the density of ch.-scalar energy can be represented in the form $w^\sigma = \theta^{\alpha\sigma}{}_{,\alpha}$, where the superpotential is equal to

$$\theta^{\alpha\sigma} = \frac{1}{\sqrt{g_{00}}} \mathfrak{M}_0^{\alpha\sigma} + 2\left(\frac{1}{\sqrt{g_{00}}}\right)_{,\beta} \cdot \mathfrak{N}_0^{\alpha\beta\sigma}$$

(for the complete physical system). This superpotential is not difficult to antisymmetrize according to the general rule:

$$\bar\theta^{\alpha\sigma} = -\bar\theta^{\sigma\alpha} = \frac{1}{2\sqrt{g_{00}}} \left(\mathfrak{M}_0^{\alpha\sigma} - \mathfrak{M}_0^{\sigma\alpha}\right) - \frac{1}{3\sqrt{g_{00}}} \left(\mathfrak{M}_0^{\alpha\beta\sigma} - \mathfrak{M}_0^{\sigma\beta\alpha}\right)_{,\beta} + \left(\frac{2}{3\sqrt{g_{00}}}\right)_{,\beta} \left(\mathfrak{M}_0^{\alpha\beta\sigma} - \mathfrak{M}_0^{\sigma\beta\alpha}\right),$$

so that $w^{\sigma} = \theta^{\alpha\sigma}_{,\alpha} = \bar{\theta}^{\alpha\sigma}_{,\alpha}$. Using invariant Lagrangians in these expressions, we obtain the definitions of the energies of fields (in particular, of the gravitational field) that are chronometrically invariant and correspond to these approaches.

As is known, the first clear formulation of the requirements imposed on the energy-momentum "complex" in the general theory of relativity (4 conditions) is due to Møller [13]. The generalization of these requirements from the point of view of Zelmanov' s formalism is obvious. At the same time, however, it is necessary to introduce also the chr.invariant 3-momentum, an element of which (cf. (3)) will, of course, be a chr.object (together with the spin element). If, following Møller, one considers the island model of the Universe, then it is necessary to require that such an integral chr.invariant (not chr.scalar as a whole!) 4-momentum under a Lorentz transformation behave as a special-relativistic 4-vector. Since what is involved here are integral quantities, these transformations must be accompanied by a transition to a new simultaneity hypersurface (a hyperbolic rotation of the hypersurface of integration). From this follow the restrictions adopted by Møller on the asymptotics of the energy-momentum complex (see another point of view in Schmutzer [14]).

However, as Bohm [15] showed from considerations of relativistic causality, the definition of any characteristics of physical systems on a space-like hypersurface as a whole has no meaning. Instead, it is reasonable to take the light cone coming from the past, with its vertex at the point of observation (the hypersurface of "simultaneous visibility"). In this sense one may also understand the integration in formula (7). In view of the general invariance of the light cone, the question of rotations of the hypersurface in transitions between reference frames also disappears.

The authors express their deep gratitude to A. L. Zelmanov for a valuable discussion.

Peoples' Friendship University
named after Patrice Lumumba

Received
1 XII 1966

CITED LITERATURE

[1] A. L. Zelmanov, DAN, **107**, 815 (1956).
[2] C. Cattaneo, Nuovo Cim., **10**, 318 (1958); Rendiconti Mat., **20**, 48 (1961); **21**, 373 (1962).
[3] E. Schmutzer, Zs. Naturforsch., **19a**, 665 (1964).
[4] A. Einstein, Sitzungsber. Preuß. Akad. Wiss., 1915, 778.
[5] H. Bauer, Phys. Zs., **19**, 163 (1918).
[6] E. Schrödinger, Phys. Zs., **19**, 4 (1918).
[7] C. Møller, Ann. of Phys., **4**, 347 (1958).
[8] V. A. Fok, *The Theory of Space, Time and Gravitation*, Moscow, 1961.

[9] L. D. Landau, E. M. Lifshitz, *The Theory of Fields*, Moscow, 1960.
[10] N. V. Mitskevich, Ann. d. Phys., **1**, 319 (1958).
[11] I. D. Novikov, Vestn. MGU, No. 2, 59 (1960).
[12] N. V. Mitskevich, Dokl. Bolgarsk. AN, **14**, 439 (1961).
[13] C. Møller, Ann. of Phys., **12**, 118 (1961).
[14] E. Schmutzer, Ann. d. Phys., **14**, 319 (1964).
[15] D. Bohm, *The Special Theory of Relativity*, Amst., 1965.

Note: Figure translations are in progress. See original paper for figures.