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Abstract

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ON PERIODIC SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

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1. In papers ([1–5]), on the basis of various kinds of majorizing relations, a priori estimates and theorems on the existence of periodic solutions of nonlinear differential equations were obtained. In the present note these results are strengthened by means of the more accurate estimates, developed in ([6–8]), for the norms of the corresponding inverse operators. The main results are formulated in Sec. 2 as applied to symmetric periodic solutions of autonomous differential equations with one nonlinearity.

2. Consider the system

$$\frac{dx_k}{dt} = \sum_{s=1}^{n} a_{ks} x_s + b_k f(\sigma), \qquad \sigma = \sum_{s=1}^{n} c_{sx} s, \qquad k = 1, 2, \dots, n, \tag{1}$$

where a_{ks}, b_k, c_s are real constants, and the function $f(\sigma)$ is defined and continuous for all finite values of σ, $f(0) = 0$.

Denote

$$D(\lambda) = \det \|a_{ks} - \lambda\delta_{ks}\| \quad (\delta_{kk} = 1,\ \delta_{ks} = 0,\ k \neq s).$$

We shall investigate the existence of symmetric periodic solutions of system (1):

$$x_k(t+T) = x_k(t), \qquad x_k(t+T/2) = -x_k(t), \qquad k = 1, 2, \dots, n, \tag{2}$$

for which

$$D(ij\omega) \neq 0, \qquad \omega = 2\pi/T, \qquad j = 2l+1, \qquad l = 1, 2, \dots$$

For these solutions, choosing the origin of t in an appropriate way, we represent the argument of the nonlinear function $f(\sigma)$ in the form

$$\sigma = a \sin \psi + z(\psi),$$

where

$$\psi = \omega t, \qquad a \geq 0, \qquad \int_0^\pi z(\psi) e^{i\psi}\, d\psi = 0.$$

As shown in ([6,8]), the frequency ω, the sum of the higher harmonics $z(\psi)$, and the amplitude a of the first harmonic of the solution satisfy the system of integral equations

$$z(\psi) = \int_0^\pi R(\psi - \theta) f[a \sin \theta + z(\theta)]\, d\theta, \tag{3}$$

$$K(a, \omega, z) = 0, \tag{4}$$

where

$$R(\psi) = \frac{2}{\pi} \sum_{l=1}^{\infty} \mathrm{Re}\,[W(ij\omega) e^{ij\psi}], \qquad j = 2l + 1;$$

$$W(\lambda) = \frac{N(\lambda)}{D(\lambda)}; \qquad N(\lambda) = \det \begin{Vmatrix} a_{11} - \lambda & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} - \lambda & b_n \\ c_1 & \cdots & c_n & 0 \end{Vmatrix} = \sum_{s=1}^{n} c_s N_s(\lambda);$$

$$K(a, \omega, \varepsilon) = aD(i\omega) - \frac{2N(i\omega)}{\pi} \int_0^\pi e^{i(\pi/2 - \psi)} f[a \sin \psi + \varepsilon(\psi)]\, d\psi.$$

If all roots λ_ρ of the characteristic equation $D(\lambda) = 0$ are simple, then

$$R(\psi) = \frac{1}{\omega} \sum_{\rho=1}^{h} \frac{N(\lambda_\rho) \exp(\lambda_\rho \psi / \omega)}{D'(\lambda_\rho)(1 + \exp 0.5 \lambda_\rho T)} - \frac{2}{\pi} \mathrm{Re}[W(i\omega) e^{i\psi}], \qquad 0 < \psi < \pi.$$

If system (3), (4) has a solution, then the periodic solution (2) of system (1) is determined by the expression:

$$x_k(\psi) = \int_0^\pi R_k(\psi - \theta) f[a \sin \theta + z(\theta)]\, d\theta, \qquad k = 1, 2, \ldots, n, \tag{5}$$

where

$$R_k(\psi) = \frac{2}{\pi} \sum_{l=0}^{\infty} \mathrm{Re}[W_k(ij\omega)e^{ij\psi}], \qquad j = 2l+1, \qquad W_k(\lambda) = \frac{N_k(\lambda)}{D(\lambda)}.$$

It is not difficult to show that, by solving simple variational problems (6), one can determine continuous, for $a \geq 0,\ \omega \geq 0,\ \alpha \geq 0$, functions $K_r(a, \omega, \alpha)$, $r = 1, 2, 3, 4$, such that

$$K_1 \geq \sup_{|\varepsilon| \leq \alpha} \mathrm{Re}\, K(a, \omega, \varepsilon), \qquad K_2 \geq - \inf_{|\varepsilon| \leq \alpha} \mathrm{Re}\, K(a, \omega, \varepsilon),$$

$$K_3 \geq \sup_{|\varepsilon| \leq \alpha} \mathrm{Im}\, K(a, \omega, \varepsilon), \qquad K_4 \geq - \inf_{|\varepsilon| \leq \alpha} \mathrm{Im}\, K(a, \omega, \varepsilon),$$

where the functions $\varepsilon(\psi)$ are assumed continuous.

Theorem. *Suppose that there exists a continuous function $\alpha_0(a, \omega)$, defined in some subdomain Ω of the first quadrant of the plane a, ω, and a function $U(a, \alpha)$, defined for $a < 0,\ \alpha > 0$, such that:*

a) for arbitrary continuous functions $\varepsilon_1(\psi), \varepsilon_2(\psi),\ \psi \in [0, \pi]$, satisfying the conditions $|\varepsilon_1(\psi)| \leq \alpha,\ |\varepsilon_2(\psi)| \leq \alpha$, the inequality

$$|f(a \sin\psi + \varepsilon_1) - f(a \sin\psi + \varepsilon_2)| \leq U(a, \alpha) \sup_{\psi \in [0, \pi]} |\varepsilon_1 - \varepsilon_2|; \tag{6}$$

b) for $\alpha \geq \alpha_0(a, \omega)$ the inequality

$$\alpha \left[1 - U(a, \alpha) \int_0^{\pi} |R(\psi)|\, d\psi\right] - \sup_{\psi \in [0, \pi]} \left| \int_0^{\pi} R(\psi - \theta) f(a \sin\theta)\, d\theta \right| \geq 0; \tag{7}$$

c) the system of inequalities

$$K_r[a, \omega, \alpha_0(a, \omega)] \geq 0, \qquad r = 1, 2, 3, 4, \tag{8}$$

determines a closed subdomain $\Omega_0 \subset \Omega$, and its contour Γ consists of four simple arcs, for which respectively $K_r = 0$.

Then system (1) has at least one nonzero symmetric periodic solution (5), for which

$$(a, \omega) \in \Omega_0, \qquad |z(\psi)| \leq \sup_{(a, \omega) \in \Omega_0} \alpha_0(a, \omega). \tag{9}$$

The proof of the theorem reduces to proving the existence, under the corresponding conditions, of a solution of system (3), (4). First it is shown that in the domain Ω equation (3) has a solution $z(\psi, a, \omega)$, and then that equation (4) has a solution (a, ω) after substituting $z = z(\psi, a, \omega)$.

To determine $z(\psi, a, \omega)$, fix some values of a and ω from Ω.

and construct the process of successive approximations:

$$z_0(\psi) \equiv 0, \qquad z_{m+1}(\psi) = \int_0^\pi R(\psi - \theta) f[a \sin\theta + z_m(\theta)]\, d\theta. \tag{10}$$

If inequality (7) is satisfied, then the sequence $\{z_m(\psi, a, \omega)\}$ converges uniformly to the solution $z(\psi, a, \omega)$ of equation (3). Moreover, the function $z(\psi, a, \omega)$ is continuous in a and ω in Ω, and the estimate $|z| \leqslant \alpha_0(a, \omega)$ holds.

It follows from this that in that region of the (a, ω)-plane where (7) is satisfied, $K(a, \omega, z)$, upon substituting $z = z(\psi, a, \omega)$, forms a continuous vector field. If there now exists a region Ω_0 satisfying the conditions of the theorem, then the rotation of the field K on the contour Γ is nonzero; hence there follows the existence of at least one* solution of equation (4) for which $(a, \omega) \in \Omega_0$. In particular, by the definition of the region Ω_0, $a > 0$, so that the corresponding solution (5) of system (1) is distinct from the trivial one.

We note that for the approximate solution $\sigma_0(t) = a_0 \sin n\omega_0 t$, found by the Krylov-Bogolyubov method, the relation $K(a_0, \omega_0, 0) = 0$ is satisfied, whence $(a_0, \omega_0) \in \Omega_0$. Thus, the theorem presented above makes it possible to give an estimate of the error of the approximate solution.

3. The results obtained naturally extend to problems where the kernel R has a more general form; in particular, to nonsymmetric problems (this requires taking into account the even harmonics of the solution); to systems with periodic coefficients and other nonautonomous systems (in this case, the phase of the solution will be the unknown instead of the frequency); to systems with retarded argument, and also to systems with many nonlinearities.

In nonautonomous problems the process of successive approximations may converge directly for $\sigma(\psi)$; in such cases there is no need to single out an equation of type (4) for the first harmonic, and it is possible to prove uniqueness of the periodic solution of the given period. Conversely, it may happen that the process of successive approximations (10) diverges. In this case, to ensure convergence one has to single out m equations of type (4) for the first m harmonics. The estimates can also be sharpened by a suitable separation of the linear part of a nonlinear function of the form

$$f(\sigma) = f_*(\sigma) + q\sigma.$$

4. We give examples illustrating the effectiveness of the estimates presented.

A. For the van der Pol equation

$$d^2x/dt^2 + x = \mu(x^2 - 1)\,dx/dt$$

the existence of a periodic solution of the form $x = a\sin\omega t + x_*(t)$ follows from the theorem for $0 \leqslant \mu < 0.16$; moreover, for $\mu = 0.1$ the inequalities $|a - a_0| \leqslant 0.07$, $|\omega - \omega_0| \leqslant 0.003$, $|x_*| \leqslant 0.034$ hold, where $a_0 = 2$, $\omega_0 = 1$. The corresponding estimates obtained for the same problem in (2) have the form

$$|a - a_0| \leqslant 0.19, \quad |\omega - \omega_0| \leqslant 0.018, \quad |x_*| \leqslant 0.046.$$

B. For the nonautonomous problem

$$d^2x/dt^2 + x + \mu\varphi(x, dx/dt, t) = 0,$$

where φ is periodic in t with period T and satisfies the conditions

* The uniqueness of the solution for $(a, \omega) \in \Omega_0$ can be proved if, instead of (6), one requires the fulfillment of the corresponding inequality for the difference of second order.

$$|\varphi(0,0,t)| \leqslant \eta; \quad |\varphi'_x(0,0,t)| \leqslant \alpha;$$

$$|\varphi'_x(0,0,t)| \leqslant \alpha; \quad |\varphi''_{xx}| \leqslant L; \quad |\varphi''_{x\dot{x}}| \leqslant L; \quad |\varphi''_{\dot{x}\dot{x}}| \leqslant L,$$

the process of successive approximations converges without separating out the lower harmonics. A periodic solution of period T exists and is unique for

$$(\alpha + \sqrt{2\eta L})\mu < \mu_0,$$

where

$$\mu_0\{2[T/2\pi]{+}1{+}2[T/2\pi{+}1/2]{+}{+}\sin(T/2{-}\pi[T/\pi]){-}\cos(T/2{-}\pi[T/2\pi])\} = |\sin T/2|.$$

Here $[\beta]$ is the integer part of β. In ([3]), Theorem 5 (3.XVIII), for the same problem the estimate

$$(\alpha + \sqrt{2\eta L})\mu < \mu_1, \qquad \mu_1 = (1 - \cos T)/6.5T$$

was obtained. It is not hard to show that $\mu_0 > \mu_1$ for all $T \neq 2l\pi$. Thus, for $T < \pi$,

$$\mu_0 = 1/(1 + \operatorname{tg} T/4).$$

B. For the equation

$$d^2x/dt^2 + x^3 = \sin t$$

one can prove the existence of a periodic solution of the form

$$x = a_1 \sin t + a_3 \sin 3t + x_*(t).$$

On transforming this equation to the form

$$d^2x/dt^2 + 4x = \sin t + (4x - x^3)$$

the following relations hold:

$$U = 4; \quad \int_0^\pi |R(\psi)|\, d\psi = 0.062\ldots; \quad \left|\int_0^\pi Rf(a, \sin\theta + a_3 \sin 3\theta)\, d\theta\right| \leqslant 0.013,$$

if

$$|a_1| \leqslant 1.46, \qquad |a_3| \leqslant 0.15.$$

The estimates for the solution have the form

$$|a_1 - a_{10}| \leqslant 0.006; \quad |a_3 - a_{30}| \leqslant 0.017; \quad |x_*(t)| \leqslant 0.011,$$

where

$$a_{10} = 1.434, \ldots, \qquad a_{30} = -0.124\ldots,$$

so that

$$|x(t) - x_0(t)| \leqslant 0.034.$$

In ([4]), for the same problem, the estimates

$$|a_1 - a_{10}| \leqslant 0.070; \qquad |a_3 - a_{30}| \leqslant 0.070; \qquad |x(t) - x_*(t)| \leqslant 0.231$$

were obtained.

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REFERENCES

[1] Yu. A. Ryabov, *DAN*, **118**, No. 4 (1958).
[2] Yu. A. Ryabov, *Proceedings of the International Symposium on Nonlinear Oscillations*, **1**, Publishing House of the Academy of Sciences of the Ukrainian SSR, 1963.
[3] L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Moscow, 1959.
[4] L. Cesari, *Contrib. to diff. equat.*, **1**, No. 2 (1963).
[5] L. Cesari, *Michigan Math. J.*, **11**, No. 4 (1964).

[6] E. N. Rosenwasser, *Proceedings of the Second All-Union Congress on Theoretical and Applied Mechanics*, vol. 2, "Nauka," 1965.
[7] E. D. Garber, *Automation and Telemechanics*, **24**, No. 4 (1963).
[8] E. D. Garber, E. N. Rosenwasser, *Automation and Telemechanics*, **26**, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.