ON THE EQUATIONS OF THE GENERAL CIRCULATION OF THE UPPER ATMOSPHERE

GEOPHYSICS

1967

SovietRxiv

Abstract

Full Text

UDC 525.70+551.513

GEOPHYSICS

M. N. IZAKOV

ON THE EQUATIONS OF THE GENERAL CIRCULATION OF THE UPPER ATMOSPHERE

(Presented by Academician G. I. Petrov on 6 V 1967)

1. The structure and dynamics of the upper atmosphere, i.e., the distributions of temperature, pressure, density, composition, and winds at altitudes greater than 30–50 km, and their temporal variations, have recently been intensively studied both theoretically and experimentally in connection with a number of scientific and technical problems (see, for example, ([1], [2])). It is of interest to construct a theoretical model that would relate the structure and dynamics of the upper atmosphere to the factors determining it—above all to ultraviolet and X-ray solar radiation. In view of the complexity of the problem, the existing models are essentially semiempirical, and various simplifications were used in deriving them ([1], [3]–). In the present work the limits of applicability of a hydrodynamic description of the upper atmosphere are estimated, and it is shown that, for motions of global scale, the equations describing the structure and dynamics of the upper atmosphere can be simplified by reducing them to equations similar to the boundary-layer equations.

2. The most complete description of the upper atmosphere is possible with the apparatus of modified Boltzmann kinetic equations for a mixture of gases, taking into account their interaction with radiation, with the addition of the equation of radiative transfer ( ). Multiplying the Boltzmann equations by summational invariants and integrating over velocity space, one obtains the equations of hydrodynamics; for the concrete calculation of the terms entering into them, the Chapman–Enskog method (,) or Grad' s method (,) is used. When the Chapman–Enskog method is applied, depending on the number of terms retained in the expansion of the distribution function, one successively obtains the Euler, Navier–Stokes, and Burnett equations. Let us estimate the range of altitudes in which the equations of hydrodynamics in the Navier–Stokes form are applicable. For this purpose we take the maximum and minimum values of the structural parameters according to ([1], [2]) (attained, respectively, by day at maximum and by night at minimum solar activity) and calculate the Knudsen number $K = \lambda L^{-1}$ (the ratio of the mean free path to the characteristic scale), taking as the characteristic scale the height scale $H = R_0 T M^{-1} g^{-1}$ (where R_0

is the gas constant, T the temperature, M the molecular weight, and g the acceleration of gravity). Since the Burnett terms are related to the principal terms as K^2, then, assuming that these terms should not exceed 4%, we find that it must be $K \leq 0.2$. From Table 1 it is seen that, under minimum conditions, the Navier–Stokes equations are applicable up to an altitude of about 220 km, and under maximum conditions approximately up to 450 km.

3. Thus, for describing the upper atmosphere up to the indicated altitudes, we have the system of Navier–Stokes equations with chemical reactions, which may be written in the form ( )

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) = 0; \tag{1}$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\mathbf{v}}{\partial\mathbf{r}}\right) + \operatorname{grad} p + \operatorname{div} P - \sum_\alpha \rho_\alpha F_\alpha = 0; \tag{2}$$

$$\rho\frac{\partial(c_vT)}{\partial t} + \rho\mathbf{v}\frac{\partial(c_vT)}{\partial\mathbf{r}} + \operatorname{div}\mathbf{q} + p\ \operatorname{div}\mathbf{v} + P : \frac{\partial\mathbf{v}}{\partial\mathbf{r}} - \sum_\alpha \rho_\alpha F_\alpha \mathbf{u}_\alpha = Q - L; \tag{3}$$

$$p = nkT = \sum_\alpha p_\alpha = \sum_\alpha n_\alpha kT; \tag{4}$$

$$\partial n_\alpha/\partial t + \operatorname{div}\left[n_\alpha(\mathbf{u}_\alpha + \mathbf{v})\right] = K_\alpha; \tag{5}$$

$$\sum_\beta \frac{n_\alpha n_\beta}{n^2 D_{\alpha\beta}}(\mathbf{u}_\alpha - \mathbf{u}_\beta) = \frac{\partial}{\partial r}\left(\frac{n_\alpha}{n}\right) + \left(\frac{n_\alpha}{n} - \frac{n_\alpha m_\alpha}{\rho}\right)\frac{\partial \ln p}{\partial r} - \left(\frac{n_\alpha m_\alpha}{p\rho}\right)\left(\frac{\rho}{m_\alpha}F_\alpha - \sum_\beta n_\beta F_\beta\right) - \sum_\beta \frac{n_\alpha n_\beta}{n_\alpha D_{\alpha\beta}}\left(\frac{D^T_\alpha}{m_\alpha n} \right. \tag{6}$$

Here t is time; $\mathbf{r}$ is the radius vector; ρ, p, T are the gas density, pressure, and temperature; c_v is the heat capacity of the gas including the internal degrees of freedom; k is Boltzmann' s constant; n_α is the particle-number density of the α-th component; $n = \sum n_\alpha$; F_α is the external force acting on particles of the α-th component; $\mathbf{v}$ is the mass velocity of the gas; $\mathbf{u}_\alpha$ is the diffusion velocity of the α-th component; P is the tensor of viscous stresses with components

$$P^{ik} = \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\frac{\partial v_i}{\partial x_i}\delta_{ik}\right); \tag{7}$$

$\mathbf{q}$ is the heat-flux vector

$$\mathbf{q} = -\lambda \frac{\partial T}{\partial r} + \sum_\alpha n_\alpha h_\alpha \mathbf{u}_\alpha + \frac{kT}{n} \sum_\alpha \sum_\beta \frac{n_\beta D_\alpha^T}{m_\alpha D_{\alpha\beta}} (\mathbf{u}_\alpha - \mathbf{u}_\beta); \tag{8}$$

h_α is the enthalpy of component α, including the energy of the internal degrees of freedom; $h_\alpha = 5/2\, kT + U_\alpha^{\text{int}}$; $D_{\alpha\beta}$ are the coefficients of mutual diffusion in a binary mixture; λ, μ, D_α^T are the coefficients of thermal conductivity, viscosity, and thermal diffusion of a multicomponent mixture, expressions for which may be taken from ([8]). The amount of heat due to absorbed radiation Q may be expressed in the form (see, for example, ([3]))

$$Q(h) = \sum_\alpha n_\alpha(h) \int_0^\infty d\lambda \cdot \varepsilon_\alpha(\lambda) I_\lambda \sigma_\alpha(\lambda) \exp\left(-\sum_\alpha \int_h^\infty n_\alpha(h) \sigma_\alpha(\lambda) \sec\theta \, dh \right); \tag{9}$$

where I_λ is the spectral flux of solar radiation of wavelength λ outside the atmosphere; σ_α is the absorption cross section of radiation by the α-th component of the gas; ε_α is the efficiency of conversion of absorbed radiation into heat; θ is the solar zenith angle; h is height. L—radiation emitted by gas particles—may be expressed in the form $L = n_\alpha f(T)$ (see, for example, ([3])). The balance of the number of particles of the α-th component appearing and disappearing as a result of chemical reactions, in the description of the neutral atmosphere, where the dissociation reaction of oxygen is important, may be expressed in the form (see, for example, ([4])):

$$K_2 = -Jn_2 + \alpha n_1^2 n + \beta n_1 n_2 n, \tag{10}$$

where the indices 1, 2 refer respectively to atomic and molecular oxygen; α, β are the recombination coefficients, respectively, in the reactions of formation of molecular oxygen and ozone, which are functions of temperature; J is the dissociation rate, expressed, like Q, by (9) with ε_α replaced by δ_α, the quantum yield of dissociation.

The system of equations (1)–(6) describes the structure and dynamics of the heterosphere—the most variable region of the upper atmosphere, lying at altitudes greater than 90-110 km, where the composition changes substantially. At lower altitudes, where the composition remains unchanged (in the homosphere), the system (1)–(4) is sufficient; however, here turbulence must be taken into account (for example, by introducing coefficients of turbulent diffusion and thermal conductivity). In addition, below a certain height, instead of the simplified accounting of the interaction of the atmosphere with radiation by means of Q, L, it is necessary to apply the radiative-transfer equation. The system (1)–(6) is written for a neutral atmosphere, but it can be generalized for the joint consideration of neutral and ionized components of the atmosphere by including electromagnetic forces and taking into account the influence of the geomagnetic field.

4. In view of the considerable complexity of the system (1)–(6), it is expedient to introduce the following simplification. We shall consider the general circulation of the upper atmosphere, i.e., motions having a horizontal spatial scale of the order of the Earth' s radius R, a vertical scale of the order of the height scale, and a time scale τ of the order of a half-day. It is precisely these motions that are associated with the global distributions of the structure of the upper atmosphere and its large-scale variations—diurnal, seasonal, 27-day, and 11-year. We note that an analogous consideration for the general circulation of the troposphere was carried out by N. E. Kochin ([10]).

We use a spherical coordinate system with origin at the center of the Earth, rotating together with the Earth, denoting by r the vertical, by θ the co-latitude, and by φ the longitude. The continuity equation in this system has the form

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\cdot\rho v_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho v_\varphi)=0, \tag{11}$$

whence the vertical component of the velocity v_r is equal to

$$v_r(r)-v_r(r_0)=\frac{1}{\rho r^2\sin\theta}\int_{r_0}^{r}\left[\frac{\partial}{\partial\theta}(\rho r\sin\theta\cdot v_\theta)+\frac{\partial}{\partial\varphi}(\rho r v_\varphi)+r^2\sin\theta\frac{\partial\rho}{\partial t}\right]dr. \tag{12}$$

Taking into account that over distances R, H the macroscopic parameters vary by an order of magnitude, and also that $HR^{-1}=10^{-3}$–$10^{-2}\ll 1$ (see Table 1) and $v_l\approx R\tau^{-1}\approx 150$ m/sec, we obtain from (12), analogously to ([10]), an estimate of the ratio of the vertical and horizontal components of the velocity

$$v_r/v_l\approx H/R\sim 10^{-2}. \tag{13}$$

We note that this estimate agrees well with experimental data, according to which at altitudes of 100–200 km $v_l\approx$ 100—200 m/sec, while $v_r\approx$ 0.1—1 m/sec. The magnitude of the ratio (13) is preserved both for a vertical scale equal to the height h and for the vertical scale πR.

Let us reduce the equations of motion and energy, written in spherical coordinates, to dimensionless form, multiplying all quantities by their scales. As a result of this operation it is found that, in the equations of motion for the horizontal components of the velocity, the inertial terms are of order M^2 (where M is the Mach number); the term describing the Coriolis force is $\sim\omega\tau\mathrm{M}^2$ (where ω is the angular velocity of the Earth' s rotation); the term with the pressure gradient is ~ 1, and the terms describing viscous forces are $\sim \mathrm{M}^2\mathrm{Re}^{-1}$ (where Re is the Reynolds number), with the individual summands of these terms being multiplied by various powers of the ratio HR^{-1}, so that the largest of them are of order $\mathrm{M}^2R^2\mathrm{Re}^{-1}H^{-2}$. In the equation of motion for the vertical component of the velocity, the terms with the force of gravity and with the pressure gradient are of order 1; the inertial terms are $\sim \mathrm{M}^2HR^{-1}$, the Coriolis terms are $\sim \omega\tau\mathrm{M}^2HR^{-1}$, and the largest of the viscous terms are $\sim \mathrm{M}^2\mathrm{Re}^{-1}$. In the energy equation, the time-dependent terms and the terms describing the work of pressure forces are of order 1; the largest of the terms describing the heat flux is $\sim \mathrm{Re}^{-1}R^2H^{-2}$; and the largest of the terms describing the work of viscous forces is $\sim \mathrm{M}^2\mathrm{Re}^{-1}R^2H^{-2}$.

Table 1

h, km	K min	K max	H/R min	H/R max	M min	M max	Re min	Re max
105	10^{-3}		$1.1\cdot 10^{-3}$		0.484		$1.36\cdot 10^{7}$	
150	$1\cdot 10^{-2}$	$5\cdot 10^{-3}$	$2.79\cdot 10^{-3}$	$6.35\cdot 10^{-3}$	0.306	0.202	$4.57\cdot 10^{4}$	$4.27\cdot 10^{4}$
200	$1\cdot 10^{-1}$	$2\cdot 10^{-2}$	$4.49\cdot 10^{-3}$	$8.80\cdot 10^{-3}$	0.242	0.174	$2.73\cdot 10^{3}$	$9.52\cdot 10^{3}$
250	$4\cdot 10^{-1}$	$3\cdot 10^{-2}$	$5.5\cdot 10^{-3}$	$1.0\cdot 10^{-2}$	0.22	0.16	$4.5\cdot 10^{2}$	$3.3\cdot 10^{3}$

Using the maximum and minimum values of the structural parameters according to (2) and the value $v_1 \approx R\tau^{-1}$, we compute the values of the criteria M and Re. From Table 1 it is seen that, over the entire range of altitudes where, as we established earlier, the equations of hydrodynamics are applicable, the values M, Re, HR^{-1} are such that a number of terms in the equations of motion and energy are of order 10^{-2} or less relative to the principal terms. Neglecting these terms, we obtain the following equations:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}-\frac{v_\varphi^2\operatorname{ctg}\theta}{r}\right)=2\rho\omega\cos\theta\cdot v_\varphi-\frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{\partial}{\partial r}\left(\mu\frac{\partial v_\theta}{\partial r}\right); \tag{14}$$

$$\rho\left(\frac{\partial v_\varphi}{\partial t}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_\theta v_\varphi\operatorname{ctg}\theta}{r}\right)=-2\rho\omega\cos\theta\cdot v_\theta-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi}+\frac{\partial}{\partial r}\left(\mu\frac{\partial v_\varphi}{\partial r}\right); \tag{15}$$

$$\partial p/\partial r+\rho g=0; \tag{16}$$

$$\rho\left(\frac{\partial(c_vT)}{\partial t}+v_r\frac{\partial(c_vT)}{\partial r}+\frac{v_\theta}{r}\frac{\partial(c_vT)}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial(c_vT)}{\partial\varphi}\right)-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda\frac{\partial T}{\partial r}\right)$$
$$+\mu\left[\left(\frac{\partial v_\theta}{\partial r}\right)^2+\left(\frac{\partial v_\varphi}{\partial r}\right)^2\right]+p\left(\frac{1}{r^2}\frac{\partial(r^2v_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot v_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}\right)$$
$$+\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2kT}{n}\sum_\alpha\sum_\beta\frac{n_\beta D_\alpha^T}{m_\alpha D_{\alpha\beta}}\left(u_{\alpha r}-u_{\beta r}\right)\right]+\sum_\alpha\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2n_\alpha h_\alpha u_{\alpha r}\right)=Q-L. \tag{17}$$

The term in (17) with the thermal-diffusion coefficient D_α^T is significant only for the light components—helium and hydrogen. The last two terms in (17) are also small, but still greater than 10^{-2}. The inertial terms in (14), (15) are more significant at altitudes of 100-120 km, where $M^2 \approx 0.1$–0.25, and less significant for $h > 150$ km, where $M^2 \approx 0.04$–0.06. The viscous terms are always small up to altitudes of 100-110 km, and above this they grow rapidly with height under minimum conditions, reaching $5 \cdot 10^{-2}$ already near 120 km, and more slowly under maximum conditions, reaching the same value at about 215 km. These estimates indicate that the wind in the thermosphere differs from the geostrophic wind.

5. The equations obtained differ from the original ones not only by having fewer terms. The main simplification consists in the fact that equations (14)–(17) are parabolic, similar to the boundary-layer equations, and consequently are easier to solve than the original Navier–Stokes equations. The difference from the boundary-layer equations is that, instead of constancy of pressure across the predominant direction of motion, there is, as is seen from (16), an approximately exponential variation of pressure in the vertical. Equation (16) is nothing other than the barometric formula; moreover, as a comparative estimate of the terms shows, it remains valid not only for global motions but also for motions on a scale of tens of kilometers.

Received
5 V 1967

CITED LITERATURE

1. COSPAR International Reference Atmosphere, 1965.
2. F. S. Johnson, in: *Near-Earth Space*, Moscow, 1966.
3. I. Harris, W. Priester, *J. Atmosph. Sci.*, **19**, 286 (1962).
4. F. D. Colegrove, F. S. Johnson, W. B. Hanson, *J. Geophys. Res.*, **71**, 2227 (1966).

5. J. E. Geisler, *J. Atmosph. Terr. Phys.*, **28**, 703 (1966).

6. A. I. Ivanovskii, A. I. Repnev, E. G. Shvidkovskii, *Kinetic Theory of the Upper Atmosphere*, Moscow, 1967.

7. S. Chapman, T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Moscow, 1960.

8. J. Hirschfelder, C. Curtiss, R. Bird, *Molecular Theory of Gases and Liquids*, Moscow, 1961.

9. G. Gred, *Mechanics*, **4**, 51; **5**, 61 (1952).

10. N. E. Kochin, *Transactions of the Main Geophysical Observatory*, **4**, 21 (1936); *Collected Works*, vol. 1, Moscow, 1949, p. 286.

Note: Figure translations are in progress. See original paper for figures.