ON EXTRACTING A SQUARE ROOT FROM ANTICOMMUTING SPINORS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 512.86:519.46

MATHEMATICS

S. S. SANNIKOV

ON EXTRACTING A SQUARE ROOT FROM ANTICOMMUTING SPINORS

(Presented by Academician A. I. Mal' tsev on 10 III 1966)

1. There is an elegant algebraic device which, for the group $SO(n)$ preserving the nondegenerate symmetric bilinear form

$$(x, y) = x_1y_1 + x_2y_2 + \cdots + x_{ny}n,$$

allows one to construct its covering group.* This device goes back to Dirac ([1]) (it is sometimes called extracting the square root from a vector ([2])) and consists in associating with the vector $x = (x_1, \ldots, x_n)$ an element $\hat{x} = x_i\gamma_i$ such that

$$\hat{x}^2 = (x_i\gamma_i)^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Here the γ_i must satisfy the relations

$$\gamma_i^2 = 1, \qquad \gamma_i\gamma_k + \gamma_k\gamma_i = 0 \quad (i \neq k).$$

The elements $\hat{x} = x_i\gamma_i$ form an n-dimensional subspace $\hat{X}_n$ of the Clifford algebra C_n of dimension 2^n, n even (or 2^{n-1}, n odd). The automorphisms in C_n that preserve the subspace $\hat{X}_n$ invariant,

$$S\hat{x}S^{-1} = \hat{x}' \tag{1}$$

and the scalar square $\hat{x}^2 = (\hat{x}')^2$, where $x' = Ox$, $O \in SO(n)$, form the group $\mathrm{Spin}(n)$.

Theorem 1. *The group* $\mathrm{Spin}(n)$ *is locally isomorphic to the group* $SO(n)$ *and covers it twice.*

Relation (1) defines a projective representation of the group $SO(n)$. The elements $S \in \mathrm{Spin}(n)$, as matrices $2^{n/2} \times 2^{n/2}$, n even (or $2^{(n-1)/2} \times 2^{(n-1)/2}$, n odd), realize a two-valued representation of the group $SO(n)$ in a $2^{n/2}$- (or $2^{(n-1)/2}$-) dimensional spinor space ([2]).

2. A group exhibiting a close analogy with $SO(n)$ is the symplectic group $\mathrm{Sp}(2m)$ of transformations of a $2m$-dimensional space Φ^{2m} preserving the skew-symmetric bilinear form $[\varphi, \chi]$, $\varphi, \chi \in \Phi^{2m}$. In canonical form

$$[\varphi, \chi] = \sum_{\substack{\alpha,\beta=1,2\\ i,j=1,\dots,m}} \delta_{ij}\varepsilon_{\alpha\beta}\varphi_i^\alpha \chi_j^\beta, \tag{2}$$

where

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

δ_{ij} is the Kronecker symbol.

We wish to transfer the construction described above to the group $\mathrm{Sp}(2m)$.**

* We also have in mind the case of an indefinite metric form

$$(x, y) = x_1y_1 + \dots + x_{py}p - x_{p+1}y_{p+1} - \dots - x_{ny}n$$

(the group $O(p, n-p)$, $p \leq n/2$).

** In the case when the group $\mathrm{Spin}(n) \subset \mathrm{Sp}(2^{n/2})\ \mathrm{Sp}(2^{(n-1)/2})$ and preserves a skew-symmetric bilinear form, we shall call the construction under consideration extraction of the square root from a spinor. It is known ([3]) that, for even n, the skew-symmetric form exists if the numbers $n(n-2)/8$ or $n(n+2)/8$ are odd; for odd n, if $(n-1)(n-3)/8$ or $(n^2-1)/8$ are odd numbers.

Remark. In the case of a skew-symmetric form, extraction of the square root is possible if Φ^{2m} is defined over a Grassmann algebra*, i.e., Φ^{2m} is a $2m$-dimensional subspace of the Grassmann algebra G_{2m}, generated by the basis elements φ_i^α $(\alpha = 1, 2;\ i = 1, \dots, m)$: $\varphi_i^\alpha\varphi_j^\beta + \varphi_j^\beta\varphi_i^\alpha = 0$**.

In this case the quadratic form $[\varphi, \varphi] \neq 0$. To each $\varphi = (\varphi_1^1, \varphi_2^2, \dots, \varphi_m^1, \varphi_m^2)$ we assign the element

$$\hat{\varphi} = \sum_{\substack{\alpha=1,2\\ i=1,\dots,m}} \varphi_i^\alpha a_\alpha^i \tag{3}$$

such that

$$\hat{\varphi}^2 = [\varphi, \varphi] = 2(\varphi_1^1\varphi_1^2 + \dots + \varphi_m^1\varphi_m^2). \tag{4}$$

The elements a_α^i must satisfy the relations

$$a_\alpha^i a_\beta^j - a_\beta^j a_\alpha^i = 2\varepsilon_{\alpha\beta}\delta_{ij}. \tag{5}$$

Lemma. The a_α^i generate an algebra isomorphic to the algebra $(qp)_m$ of the m-dimensional coordinate and momentum operators of quantum mechanics.

Put $a_1^i = \sqrt{-2}\,p_i,\ a_2^i = \sqrt{2}\,q_i$. Then from (5) it follows that

$$q_i p_j - p_j q_i = \sqrt{-1}\delta_{ij}, \qquad q_i q_j - q_j q_i = p_i p_j - p_j p_i = 0. \tag{5'}$$

Denote by $\Phi_{2m} \subset (q,p)_m$ the $2m$-dimensional subspace generated by monomials of the form $\hat{\varphi} = \varphi_i^\alpha a_\alpha^i$. In $(q,p)_m$ consider automorphisms preserving the invariant subspace Φ_{2m},

$$T\hat{\varphi}T^{-1} = \hat{\varphi}', \tag{6}$$

where $\varphi' = S\varphi,\ S \in \mathrm{Sp}(2m)$. These automorphisms preserve the symplectic square: $\hat{\varphi}^2 = (\hat{\varphi}')^2$.

In the case of the group $\mathrm{Sp}(2m)$, the analogue of Theorem 1 is

Theorem 2. The automorphisms (6) form a group locally isomorphic to the group $\mathrm{Sp}(2m)$.

Proof. Consider (6) in infinitesimal form. We have $S = 1 + I_\xi\theta_\xi,\ T = 1 + L_\xi\theta_\xi$, where θ_ξ are real parameters of an infinitesimal transformation from the group $\mathrm{Sp}(2m)$, and $I_\xi,\ L_\xi$ are infinitesimal operators. The elements $(I_\xi)_{ik\beta}^\alpha$ satisfy the relations

$$(I_\xi)_{ik\beta}^\alpha \varepsilon_{\alpha\gamma} + \varepsilon_{\beta\lambda}(I_\xi)_{ki\gamma}^\lambda = 0,$$

and, moreover,

$$I_\xi I_\eta - I_\eta I_\xi = C_{\xi\eta}^\zeta I_\zeta, \tag{7}$$

where $C_{\xi\eta}^\zeta$ are the structure constants of the Lie algebra of the group $\mathrm{Sp}(2m)$. Equation (6) now gives

$$L_\xi a_\alpha^i - a_\alpha^i L_\xi = (I_\xi)_{ik\alpha}^\beta a_\beta^k. \tag{8}$$

From (8) for L_ξ we obtain

$$L_\xi = \frac{1}{4}(I_\xi)_{ik\alpha}^\beta a_\beta^k a_i^\alpha \varepsilon^{\alpha\gamma}, \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta_\beta^\alpha, \tag{9}$$

where δ_β^α is the Kronecker symbol. Using the permutation relations for a_α^i (5), we establish that

$$L_\xi L_\eta - L_\eta L_\xi = C_{\xi\eta}^\zeta L_\zeta \tag{10}$$

with the $C^{\zeta}_{\xi\eta}$ from (7). Thus, the local isomorphism of T and $\mathrm{Sp}(2m)$ is established. The finite transformations T are expressed through infinitesimal operators in the form of formal series:

$$T = \sum_{n=0}^{\infty} \frac{1}{n!}(L_\xi \theta_\xi)^n = \exp(L_\xi \theta_\xi).$$

With such T, formula (6) defines a projective representation of the group $\mathrm{Sp}(2m)$. The T do not always exist as affine transformations in Hilbert space.

* Recall that in the case of a symmetric form, extraction of the square root is possible in any (Euclidean) field (see (2), p. 366).

** The scheme developed is not abstract. Spinors of quantum field theory are quantities of precisely this (algebraic) nature.

Theorem 3. The operators T define a unitary representation in Hilbert space if the numbers $(I_\xi)^1_{ik1}, (I_\xi)^2_{ik2}$ are real, and $(I_\xi)^1_{ik2}, (I_\xi)^2_{ik1}$ are imaginary.

The proof follows from definition (9). Indeed, the operators q_i, p_i (5′) can be realized as Hermitian operators in Hilbert space (Stone–von Neumann theorem). Then, under the formulated conditions, the operators

$$H_\xi = -\sqrt{-1}L_\xi = \frac{1}{2}\big[(I_\xi)^1_{ik1} p_{kq} i - (I_\xi)^2_{ik2} q_{kp} i - \\ -\sqrt{-1}(I_\xi)^1_{ik2} p_{ip} k - \sqrt{-1}(I_\xi)^2_{ik1} q_{iq} k\big]$$

are Hermitian, and, consequently, the operators $T = \exp(\sqrt{-1}H_\xi \theta_\xi)$ are unitary. (We note that H_ξ have the form of the dynamical operators of an m-dimensional oscillator.)

For the simply connected group $\mathrm{Sp}(2m, C)$ the conditions of the theorem are not satisfied. They may hold for some noncompact nonsimply connected subgroup of the group $\mathrm{Sp}(2m, C)$. In particular, the conditions of the theorem are satisfied for the real symplectic group $\mathrm{Sp}(2m, R)$. In this case the operators T form a group which doubly covers the group $\mathrm{Sp}(2m, R)$.* In Hilbert space the operators T define a two-valued representation of the group $\mathrm{Sp}(2m, R)$. In the realization $q_i = x_i,\ p_i = \dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x_i}$ $(-\infty < x_i < \infty)$, the representation in the class $L_2(m)$ of quadratically integrable functions $\int f|x_1, \dots, x_m|^2 dx_1 \dots dx_m < \infty$ is defined by the formula

$$T_s f(x) = T_s f(x) T_s^{-1}(T_s \cdot 1) = f(T_{sxT} s^{-1})\chi(x; s) = f(xs)\chi(x; s) \\ s \in \mathrm{Sp}(2m, R), \tag{11}$$

where $\chi(x; s) = T_s \cdot 1$, and

$$(xs)_i = s^2_{ij2} x_j + s^1_{ij2} \frac{\partial}{\partial x_j}.$$

The group condition leads to the following equation for $\chi(x;s)$:

$$T_{s_2}\chi(x;s_1) = \chi(xs_2;s_1)\chi(x;s_2) = \chi(x;s_2 s_1) \quad (12)$$

with boundary condition $\chi(x;1) = 1$.

In (4) such representations of the group $\mathrm{Sp}(2,R)$ are constructed explicitly (one-dimensional oscillator).

3. Our main result consists in the fact that the operation of extracting the square root is solvable at least for doubly connected symplectic groups.** We can transfer this operation also to simply connected symplectic groups, for example $\mathrm{Sp}(2m,C)$. Then extraction of the root leads to infinite-dimensional spaces of a more general type than Hilbert spaces. Indeed, now the operators $\sqrt{-1}L_\xi$ are not Hermitian, and, consequently, T are not unitary.

The representation can be obtained from (11), (12) by complexifying $s \in \mathrm{Sp}(2m,R)$, as a result of which we obtain the group $\mathrm{Sp}(2m,C)$, and x_i (now the transformations (6) are complex) also become complex. In this case the functions $f(x)$ (x_i complex) become elements of an infinite-dimensional space with a strongly indefinite metric. For the group $U\,\mathrm{Sp}(2)$ this question was partially discussed in (5).

Physical-Technical Institute
Academy of Sciences of the Ukrainian SSR

Received
12 II 1966

CITED LITERATURE

1. P. A. M. Dirac, Proc. Roy. Soc., **A117**, 610 (1928).

2. H. Weyl, *Classical Groups*, IL, 1947.

3. P. K. Rashevskii, UMN, **10**, no. 2 (1955).

4. S. S. Sannikov, ZhETF, **49**, no. 6 (1965).

5. S. S. Sannikov, Ukr. Phys. Zh., **10**, no. 6 (1965); Yadern. Fiz., **2**, no. 3 (1965).

* The group space of the group T is a $(4m^2 - 1)$-dimensional manifold in Hilbert space.

** For orthogonal groups the extraction of the root is always possible (and leads to finite-dimensional spaces), since all orthogonal groups are at least doubly connected.

Note: Figure translations are in progress. See original paper for figures.