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Abstract

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PRENUCLEAR MAPPINGS IN SCALES OF BANACH AND HILBERT SPACES

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Let E and F be two Banach spaces with unit balls $S_1(E)$ and $S_1(F)$, respectively. Denote by $S_1(E)^0$ the unit ball of the conjugate space E'. A continuous linear mapping $y = T(x)$, acting from the space E into F, is called **prenuclear** (see $(^1)$) if on $S_1(E)^0$ there exists a positive Radon measure μ such that

$$\|Tx\| \le \int_{S_1(E)^0} |\langle x, u\rangle| \, d\mu \quad \text{for all } x \in E \ (u \in E'). \tag{1}$$

The prenuclear norm $\pi(T)$ for this mapping is defined by the formula

$$\pi(T) = \inf \|\mu\|,$$

where the infimum is taken over all measures μ satisfying condition (1).

By the symbol $L^1(E)$ we shall everywhere in what follows denote the Banach space of absolutely convergent sequences $\bar{x} = (x_1, x_2, \dots, x_n, \dots)$ of elements of the space E with norm

$$\|\bar{x}\|_{L^1(E)} = \sum_{k=1}^{\infty} \|x_k\|_E,$$

and by $l^1(E)$ we denote the space of all unconditionally convergent sequences $\bar{x} = (x_1, x_2, \dots, x_n, \dots)$ $(x_n \in E)$, composed of elements of E, where

$$\|\bar{x}\|_{l^1(E)} = \sup_{u \in S_1(E)^0} \sum_{k=1}^{\infty} |\langle x_k, u\rangle|.$$

It is said that the mapping $y = T(x)$ $(y \in F,\ x \in E)$ is **absolutely summing** if T maps every unconditionally convergent sequence $x_n \in E$ into an absolutely convergent one.

In (^{1}) it is shown that the class of all prenuclear mappings $y = T(x)$ $(y \in F,\ x \in E)$ coincides with the set of absolutely summing mappings; moreover

$$\pi(T) = \sup \sum_{k=1}^{\infty} \|Tx_k\|_F,$$

where the supremum is taken over all elements $x_k \in E$ for which

$$\sup_{u \in S_1(E)^0} \sum_{k=1}^{\infty} |\langle x_k, u\rangle| \leq 1.$$

It follows from this that every prenuclear mapping $T : E \to F$ generates a mapping $\overline{T} : l^1(E) \to L^1(F)$ by the formula

$$\bar{y} = \overline{T}(\bar{x}) = \{Tx_k\}_{k=1}^{\infty},$$

and the prenuclear norm $\pi(T)$ is the usual norm of the mapping $\overline{T}$.

Let F_α $(0 \leq \alpha \leq 1)$ be an arbitrary normal scale of Banach spaces (see (2)). It is easy to see that in this case the family of spaces $L^1(F_\alpha)$ also forms a normal scale.

Proposition 1. If $F_\alpha^{\min}$ $(0 \leq \alpha \leq 1)$ is the minimal scale (see (2)) joining the spaces F_0 and F_1, then the scale $L^1(F_\alpha^{\min})$ majorizes (see (2)) the minimal scale constructed from the spaces $L^1(F_0)$ and $L^1(F_1)$.

Proposition 2. The family of spaces $L^1(F_\alpha^{\max})$ forms the maximal scale (see (2)) joining the space $L^1(F_0)$ with $L^1(F_1)$, if the scale $F_\alpha^{\max}$ is maximal.

Remark 1. It follows from Propositions 1 and 2 that the scale $L^1(F_\alpha^{\min})$ has the normal interpolation property for linear operators (see (2)) with respect to any minimal scale, while the scale $L^1(F_\alpha^{\min})$ is strictly interpolation with respect to any normal scale. We note that the spaces $L^1(F_\alpha^{\min})$ do not necessarily form a minimal scale: it may happen that the scale $L^1(F_\alpha^{\min})$ is maximal.

Proposition 3. For any normal scale E_α $(0 \leq \alpha \leq 1)$, the spaces $l^1(E_\alpha)$ form a normal scale.

Proof. Indeed,

$$\varphi(\alpha) = \|\bar{x}\|_{l^1(E_\alpha)} = \sup_{u \in S_1(E_\alpha)^0} \sum_{k=1}^{\infty} |\langle x_k, u\rangle| =$$

$$= \sup_{\substack{|\theta_k|=1 \\ u\in S_1(E_\alpha)^0}} \sum_{k=1}^{\infty} \langle \theta_k x_k, u\rangle = \sup_{\substack{|\theta_k|=1 \\ u\in S_1(E_\alpha)^0}} \left\langle \sum_{k=1}^{\infty} \theta_k x_k, u \right\rangle = \sup_{|\theta_k|=1} \left\| \sum_{k=1}^{\infty} \theta_k x_k \right\|_{E_\alpha} .$$

Hence the logarithmic convexity of the function $\varphi(\alpha)$ follows, since the supremum of logarithmically convex functions is a logarithmically convex function.

The remaining axioms of a normal scale are obvious.

Theorem 1. A normal scale of Banach spaces $l^1(E_\alpha)$ $(0 \le \alpha \le 1)$ is regular if and only if the family of spaces E_α $(0 \le \alpha \le 1)$ forms a regular scale (see (2)).

The proof of Theorem 1 is based on properties of Pietsch products.

Definition 1. Let $E_1, E_2, \dots, E_n$ be subspaces of a Banach space E. The **Pietsch product** $\pi(E_1, \dots, E_n)$ of the spaces $E_1, \dots, E_n$ is the direct product $E_1 \times \dots \times E_n$, endowed with the norm

$$\|\tilde{x}\|_{\pi(E_1 \dots E_n)} = \|(x_1, \dots, x_n)\|_{\pi(E_1 \dots E_n)} = \sup_{\theta_k^1} \left\| \sum_{k=1}^{\infty} \theta_k x_k \right\|_{E} ,$$

where $\tilde{x} \in E_1 \times \dots \times E_n$ and $x_k \in E_k$ $(k = 1, \dots, n)$.

In particular, if $E_1 = E_2 = \dots = E_n = E$, then the space $\pi(E_1 \dots E_n) = \pi(E^n)$ is called the n-th Pietsch power of the Banach space E.

It is not difficult to note that for any normal scale E_α $(0 \le \alpha \le 1)$, the family of spaces $\pi(E_\alpha^n)$ forms a normal scale, which we shall call the n-th Pietsch power of the scale E_α.

Lemma 1. The Pietsch square $\pi(E_\alpha^2)$ of a regular scale E_α $(0 \le \alpha \le 1)$ is a regular scale.

Proof. The unit ball $S_1[\pi(E_\alpha^2)]^0$ of the conjugate space $\pi(E_\alpha^2)'$ is closed in the topology $\sigma(\pi(E_\alpha^2)', \pi(E_\alpha^2))$

by the convex hull of the sets

$$\mathcal{U} = \{(u, u) : \ \|u\|_{E'_\alpha} = 1\}, \qquad \mathcal{V} = \{(v, -v) : \ \|v\|_{E'_\alpha} = 1\},$$

since the ball $S_1[\pi(E_\alpha^2)]^0$ coincides with the polar of the set

$$S_1[\pi(E_\alpha^2)] = \left\{ \tilde{x} = (x_1, x_2) : \sup_{|\theta_k|=1} \|\theta_1 x_1 + \theta_2 x_2\|_{E_\alpha} \leqslant 1; \ x_1, x_2 \in E_\alpha \right\}$$
$$= \left\{ \tilde{x} : \sup_{u \in S_1(E_\alpha)^0} [|\langle x_1, u\rangle| + |\langle x_2, u\rangle|] \leqslant 1 \right\} .$$

Let us show that all boundary points $\widetilde{w}_2$ of the set $S_1[\pi(E_\alpha^2)]^0$ can be represented in the form

$$\widetilde{w}_2 = \mu\tilde{u}_2 + \nu\tilde{v}_2 \qquad (\mu + \nu = 1;\ \mu, \nu \geqslant 0), \tag{2}$$

where $\tilde{u}_2$ and $\tilde{v}_2$ run over the boundaries of the sets $\mathcal{U}$ and $\mathcal{V}$, respectively. To this end consider the subspaces

$$\mathcal{L} = \{(u, u):\ u \in E'_\alpha\} \subset \pi(E_\alpha^2)', \qquad \mathcal{M} = \{(v, -v):\ v \in E'_\alpha\} \subset \pi(E_\alpha^2)',$$

each of which is isometrically conjugate to the space E'_α, for the sets $\mathcal{U}$ and $\mathcal{V}$ are the unit balls in the subspaces $\mathcal{L}$ and $\mathcal{M}$. The subspaces $\mathcal{L}$ and $\mathcal{M}$, being the polars of the subspaces $\{\tilde{x}:\ (x, -x),\ x \in E_\alpha\}$ and $\{\tilde{y}:\ (y, y),\ y \in E_\alpha\}$, are closed in the topology $\sigma(E'_\alpha \times E'_\alpha, E_\alpha \times E_\alpha)$. The unit ball of the conjugate space is a compact set in the weak topology; therefore the convex hull $\mu\mathcal{U} \oplus \mathcal{V}$ $(\mu + \nu = 1,\ \mu, \nu \geqslant 0)$ is compact in the topology $\sigma(E'_\alpha \times E'_\alpha, E_\alpha \times E_\alpha)$, so that $S_1[\pi(E_\alpha^2)]^0 = \mu\mathcal{U} \oplus \nu\mathcal{V}$ (see (3), Chapter II, §4, Proposition 1).

Consequently, to prove the representation (2) it is enough for us to show that the point $\mu\tilde{u}_2 + \nu\tilde{v}_2$ $(\mu + \nu = 1;\ \mu, \nu \geqslant 0)$ will be a boundary point of the ball $S_1[\pi(E'_\alpha)]^0$.

Choose an arbitrary number $\theta > 1$, as close to one as desired. The intersection of the subspaces $\mathcal{L}$ and $\mathcal{M}$ contains only one zero element; in view of this circumstance there exist two elements $\tilde{x}_0, \tilde{y}_0 \in E_\alpha \times E_\alpha$ for which the conditions

$$\begin{gathered} \langle \tilde{x}_0, \theta\tilde{u}_2 \rangle > 1, \qquad \langle \tilde{x}_0, \tilde{v} \rangle = 0 \quad \text{for all } \tilde{v} \in \mathcal{M}, \\ \langle \tilde{y}_0, \theta\tilde{v}_2 \rangle < 1 \qquad (\tilde{y}_0, \tilde{u}_0) = 0 \quad \text{for all } \tilde{u} \in \mathcal{L}, \\ \|\tilde{x}_0\|_{\pi(E_\alpha^2)} = \|\tilde{y}_0\|_{\pi(E_\alpha^2)} = 1. \end{gathered}$$

Consider the hyperplane $\langle \tilde{x}_0 + \tilde{y}_0, \tilde{u} \rangle = 1,\ \tilde{u} \in \pi(E_\alpha^2)'$, which is, obviously, closed in the weak topology; therefore the weakly closed convex hull of the sets $\mathcal{U}$ and $\mathcal{V}$ lies on one side of this hyperplane. Denote by l the straight line belonging to the two-dimensional plane P generated by the elements $\tilde{u}_2, \tilde{v}_2$ and passing through the points $\theta\tilde{u}_2, \theta\tilde{v}_2$. Let $\widetilde{w}$ be an arbitrary point of the segment joining the points $\tilde{u}_2$ and $\tilde{v}_2$, and let W be a neighborhood of the point $\widetilde{w}$ in the plane P. This neighborhood, for some $\theta > 1$, contains points of the straight line l_0; hence $\widetilde{w}$ is a boundary point of $S_1[\pi(E_\alpha^2)]^0$.

Now consider an arbitrary element $\tilde{u} \in \pi(E_0^2)'$. Let $\tilde{u}_2$ and $\tilde{v}_2$ be two elements of the sets $\pi(E_\alpha^2)' \cap \mathcal{U}$, $\pi(E_0^2)' \cap \mathcal{V}$, such that $\tilde{u} = \mu\tilde{u}_2 + \nu\tilde{v}_2$ $(\mu + \nu = 1;\ \mu, \nu = 0)$.

On the basis of the preceding arguments we may assert the validity of the equality

$$\|\tilde{u}\|_{\pi(E_\alpha^2)'} = \mu\|\tilde{u}_2\|_{\pi(E_\alpha^2)'} + \nu\|\tilde{v}_2\|_{\pi(E_\alpha^2)'},$$

for every $\alpha \in [0,1]$. The function $f(\alpha) = \|\tilde{u}\|_{\pi(E_\alpha^2)'}$ will be logarithmically convex as the sum of two logarithmically convex functions.

Lemma 2. The n-th power of Pisier $\pi(E_\alpha^n)$ of a regular scale E_α $(0 \le \alpha \le 1)$ is a regular scale.

Using the almost interpolation property of regular scales (see (2)) and Theorem 1, we obtain that the following is valid.

Theorem 2. Let $y = T(x)$ be a prenuclear operator mapping the spaces E_0 into F_0 and E_1 into F_1; let E_α $(0 \le \alpha \le 1)$ be a regular scale connecting the space E_0 with E_1. Then, for an arbitrary normal scale F_α $(0 \le \alpha \le 1)$, the operator $y = T(x)$ is a prenuclear mapping acting from the space E_β into F_α for $\beta > \alpha$.

A prenuclear mapping acting in a Hilbert space is a Hilbert–Schmidt mapping (see (1)).

Theorem 3. Let H_α $(0 \le \alpha \le 1)$ be a Hilbert scale (see (2)) and G_α $(0 \le \alpha \le 1)$ an arbitrary scale of Hilbert spaces. If $y = A(x)$ is a Hilbert–Schmidt operator acting from the spaces H_0 into G_0, H_1 into G_1, then A will be a Hilbert–Schmidt operator mapping the space H_α into G_α, and the Hilbert–Schmidt norm $|A|_{H_\alpha \to G_\alpha}$ is logarithmically convex as a function of α.

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Note: Figure translations are in progress. See original paper for figures.