DUAL PROBLEMS OF CONVEX AND FRACTIONAL-CONVEX PROGRAMMING IN FUNCTIONAL SPACES

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DUAL PROBLEMS OF CONVEX AND FRACTIONAL-CONVEX PROGRAMMING IN FUNCTIONAL SPACES

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The paper gives a general scheme for forming the dual problem for functional analogues of problems of convex and fractional-convex programming and formulates a number of assertions constituting the basis of the theory of duality for these problems. The proposed scheme for constructing the dual problem is analytic in character and generalizes the approach set forth in (1) for finite-dimensional problems of convex programming. We note that another, geometric principle for constructing the dual problem is contained in (2).

1. Let E and E_1 be real B-spaces; G and G_1 nonempty convex subsets of E and E_1, respectively; G_1 a cone; $f(x)$ a concave real functional defined on G; $\Phi(x)$ an operator from G into E_1, concave (relative to G_1) on G, i.e. satisfying the condition

$$\Phi(\alpha x_1 + (1-\alpha)x_2) - \alpha\Phi(x_1) - (1-\alpha)\Phi(x_2) \in G_1;$$
$$x_1, x_2 \in G, \quad 0 \le \alpha \le 1.$$

We turn E_1 into a partially ordered set by means of the condition

$$y' \le y''(G_1), \quad \text{if } y'' - y' \in G_1 \quad (y', y'' \in E_1);$$

G_1 is sometimes called the positive cone in E_1. The positive cone G_1 induces in E_1^* the convex cone $G_1^* = \{\lambda : \lambda(y) \ge 0, \text{ if } y \ge 0(G_1), \ \lambda \in E_1^*\}$, which is taken as the positive cone of this space.

The problem of maximizing the functional $f(x)$:

$$f(x) \to \sup \tag{1}$$

under the conditions

$$\Phi(x) \geq 0 \ (G_1), \tag{2}$$

$$x \in G, \tag{3}$$

is naturally called a **convex programming problem** in a functional space. Let $R = \{x : \Phi(x) \geq 0(G_1),\ x \in G\}$. We shall agree to call points $x \in R$ **plans** of problem (1)–(3); a sequence $X = \{x^{(k)}\}$ of plans $x^{(k)}$ of problem (1)–(3), for which there exists

$$\lim_{k\to\infty} f(x^{(k)}) = f(X),$$

a **plan-sequence** of this problem. A plan-sequence $X = \{x^{(k)}\}$ of problem (1)–(3) will be called a **solution** of the given problem if $f(X) = v$, where

$$v = \sup_{x\in R} f(x).$$

In particular, a stationary sequence $\{x^{(k)}\} = \{x^*\}$ may turn out to be a solution of problem (1)–(3); the point x^* will be called a **solution-plan** of problem (1)–(3).

Put

$$F(x,\lambda) = f(x) + \lambda(\Phi(x)), \quad x \in G, \quad \lambda \in E_1^*;$$

$F(x,\lambda)$ is usually called the **Lagrange functional** of problem (1)–(3).

Let

$$\varphi(x) = \inf_{\lambda \geqslant 0\ (G_1^*)} F(x,\lambda). \tag{4}$$

Consider the problem

$$\varphi(x) \to \sup \tag{5}$$

under the condition

$$x \in G. \tag{6}$$

It is not difficult to verify that, in the case when G_1 is closed, problems (1)–(3) and (4)–(6) are equivalent (the concavity of f, Φ and the convexity of G

are not essential here). The structure of problem (4)—(6), which is equivalent to the original problem (1)—(3), suggests a natural formulation for the dual problem. Problem (4)—(6) consists in first applying to the Lagrange functional $F(x, \lambda)$ the operation inf with respect to $\lambda \in G_1^*$, and then the operation sup with respect to the elements $x \in G$. In order to pass to the dual problem, we change the order in which these operations are applied.

Let

$$\psi(\lambda) = \sup_{x \in G} F(x, \lambda). \tag{7}$$

We define the problem dual to (1)—(3) as follows: it is required to find

$$\tilde{v} = \inf \psi(\lambda) \tag{8}$$

under the condition

$$\lambda \geqslant 0 \ (G_1^*). \tag{9}$$

2. A sequence $X = \{x^{(k)}\}$ of elements $x^{(k)} \in G$ will be called a **generalized plan** of problem (1)—(3) if there exists a sequence $\{y_1^{(k)}\}$ of elements $y_1^{(k)} \geqslant 0$ (G_1) such that $\Phi(x^{(k)}) = y_1^{(k)} + y_2^{(k)}$, $\lim_{k \to \infty} |y_2^{(k)}| = 0$, and, moreover, there exists $\lim_{k \to \infty} f(x^{(k)}) = f(X)$.

Let $\overline{R}$ be the set of generalized plans of problem (1)—(3). Obviously, if the generalized plan $\{x^{(k)}\} = \{x^*\}$, then x^* is a plan of problem (1)—(3). Consequently, $R \subset \overline{R}$.

By the **generalized problem** (1)—(3) we shall mean the problem of maximizing $f(X)$ on the set $\overline{R}$. Put $v' = \sup_{X \in \overline{R}} f(X)$. We agree to regard $v' = \infty$ $(v = -\infty)$ if $\overline{R} = \emptyset$ $(R = \emptyset)$. The proposition formulated below, establishing a connection between the generalized problem (1)—(3) and the dual problem (7)—(9), is naturally called the generalized duality theorem.

Theorem 1. *For an arbitrary convex programming problem (1)—(3), with* $\overline{R} \neq \emptyset$, *the generalized duality relation holds:*

$$v' = \tilde{v}. \tag{10}$$

As a consequence of Theorem 1 we obtain Theorem 2.

Theorem 2. *When* $\overline{R} \neq \emptyset$, *problem (1)—(3) and the problem dual to it (7)—(9) are connected by the duality relation:*

$$v = \tilde{v} \tag{11}$$

if and only if

$$v = v'. \tag{12}$$

In the case when the operator Φ and the functional f are linear, and G is a cone, Theorem 1 (in a somewhat different formulation) was established in (^{3}).

3. With the aid of Theorem 2, the following two duality theorems are proved. Let

$$M_\rho(f, G) = \begin{cases} \sup\limits_{x \in G \cap C_\rho} f(x), & \text{if } G \cap C_\rho \neq \emptyset, \\ -\infty, & \text{if } G \cap C_\rho = \emptyset, \end{cases}$$

where $C_\rho = \{x: \ |x| = \rho, \ x \in E\}$.

Theorem 3. *Let (1)–(3) be a convex programming problem. If*

$$\lim_{\rho \to \infty} M_\rho(f, G) = -\infty; \tag{13}$$

G and G_1 are closed sets; Φ and f are continuous on G; every bounded subset of G is weakly compact, then problems (1)–(3), (7)–(9) are connected by the duality relation (11).

Theorem 4. *Let all the conditions listed in Theorem 3 be retained, with the exception of (13). If every bounded subset of G is compact, and the set of solution-plans of problem (1)–(3) is nonempty and bounded, then problems (1)–(3) and (7)–(9) are connected by the duality relation (11).*

The conditions of Theorem 3 are satisfied, in particular, for a number of best-approximation problems with additional constraints. Theorem 4 finds application in the theory of finite-dimensional convex programming.

4. The conditions of Theorems 3 (in the case $R \neq \emptyset$) and 4 obviously guarantee the existence of a solution-plan of problem (1)–(3). However, the dual problem (7)–(9), as the simplest finite-dimensional examples show, in general does not possess this property. The conditions listed in the theorem below not only guarantee that the duality relation is fulfilled, but also ensure the existence of a solution-plan of problem (7)–(9), provided only that $\nu < \infty$. (At the same time, problem (1)–(3) may also have no solution-plan.) Let $E = E_{01} \times E_{02}$, $E_1 = E_{11} \times E_{12}$, $G = G_{01} \times G_{02}$, $G_1 = G_{11} \times G_{12}$, where $E_{\gamma i}$ ($\gamma = 0, 1, \ i = 1, 2$) is a B-space; G_{0i} is a convex subset of E_{0i}; $G_{1i} \in E_{1i}$ is a convex cone ($i = 1, 2$).

Put

$$f(x) = f(x_1, x_2); \qquad \Phi(x) = (\Phi_1(x_1, x_2), \ \Phi_2(x_1) + A(x_2)), \tag{14}$$

where $x_i \in E_{0i}$ $(i = 1, 2)$; $\Phi_1(x_1, x_2)$ is a concave (relative to G_{11}) operator acting from G into E_{11}, $\Phi_2(x_1)$ is a concave (relative to G_{12}) operator acting from G_{01} into E_{12}; $A(x_2)$ is a linear bounded operator from E_{02} into E_{12}. We shall say that the constraints of problem (1)–(3), for f and Φ defined by (14), satisfy the **generalized Slater condition** if there exists such a plan $x^* = (x_1^*, x_2^*)$ of the problem that: a) $\Phi_1(x_1^*, x_2^*)$ is an interior point of G_{11}; b) x_2^* is an interior point of G_{02}.

For $E = E_{01}$, $E_1 = E_{11}$, the condition formulated becomes the familiar Slater condition ([4,5]).

Theorem 5. *Let (1)–(3), for f and Φ defined by (14), be a convex programming problem, and let $f(x_1, x_2)$ and $\Phi(x_1, x_2)$ be continuous in $x_2 \in G_{02}$ for fixed $x_1 \in G_{01}$. If constraints (2), (3) satisfy the generalized Slater condition and the operator A maps E_{02} onto E_{12}, then problems (1)–(3) and (7)–(9) are connected by the duality relation (11), and in the case $\nu < \infty$ the lower bound (8) is attained.*

A special case of Theorem 5 $(E = E_{01}, E_1 = E_{11})$, in different terms and under somewhat stronger assumptions, is contained in ([5]).

5. Closely connected with Theorems 3–5 are the so-called optimality criteria —necessary and sufficient conditions for a certain plan-sequence of a problem of type (1)–(3) to be its solution. Similar criteria were first studied by L. V. Kantorovich as applied to linear programming problems ([6]).

Theorem 6. Let $X = \{x^{(k)}\}$ be a plan-sequence of problem (1)–(3), $f(X) < \infty$. In order that X be a solution of problem (1)–(3), it is sufficient, and in the case when the duality relation (11) is satisfied also necessary, that there exist a sequence $\{\lambda^{(k)}\}$, $\lambda^{(k)} \geqslant 0$ (G_1^*), such that

$$\lim_{k\to\infty} F(x^{(k)}, \lambda^{(k)}) = \lim_{k\to\infty} \sup_{x\in G} F(x, \lambda^{(k)}), \qquad \lim_{k\to\infty} \lambda^{(k)}(\Phi(x^{(k)})) = 0. \tag{15}$$

If, in addition to satisfaction of relation (11), the dual problem (7)–(9) has a plan-solution, then the sequence $\{\lambda^{(k)}\}$ in Theorem 6 may be taken to be stationary, i.e. $\{\lambda^{(k)}\} = \{\lambda^*\}$. Thus Theorems 3–5 single out classes of problems of the type (1)–(3) for which conditions (15) constitute an optimality criterion.

The formulations of the dual problems and optimality criteria admit refinements as applied to separate classes of problems of the type (1)–(3).

6. Let $\Phi_1(x)$ and $\Phi_2(x)$ be operators from G into the B-space E_2, and let G_2 be a convex cone in E_2. Consider the problem

$$f(x) = \inf_{\mu\geqslant 0\,(G_2^*),\ |\mu|=1} \frac{\mu(\Phi_1(x))}{\mu(\Phi_2(x))} \to \sup \tag{16}$$

under conditions (2), (3).

It is natural to call the problem (16), (2), (3) a problem of **fractional-convex programming** in a functional space, if $\Phi_1(x)$ is a concave operator with respect to G_2; $\Phi_2(x)$ is a concave (convex) operator with respect to G_2 for $v_1' > 0$ $(v_1' < 0)$,

$$v_1' = \sup_{x\in\overline{R}} f(X); \qquad \inf_{\mu\geqslant 0\,(G_2^*),\ |\mu|=1} \mu(\Phi_2(x)) \geqslant \rho > 0 \quad \text{for any } x \in G.$$

As the Lagrangian functional of problem (16), (2), (3) one takes

$$F(x,\mu,\lambda) = [\mu(\Phi_1(x)) + \lambda(\Phi(x))]/\mu(\Phi_2(x)).$$

Next put

$$\psi(\mu,\lambda) = \sup_{x\in G} F(x,\mu,\lambda) \tag{17}$$

and define the problem dual to (16), (2), (3) as follows:

$$\psi(\mu,\lambda) \to \inf, \tag{18}$$

$$\mu \geqslant 0\ (G_2^*), \qquad |\mu| = 1, \qquad \lambda \geqslant 0\ (G_1^*). \tag{19}$$

For the problem of fractional-convex programming (16), (2), (3) and the dual problem (17)–(19) to it, analogues of Theorems 1–6 hold; the formulations of the corresponding assertions for problems (1)–(3); (7)–(9) and (16), (2), (3); (17)–(19) differ only in insignificant details.

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