GEOMETRIES OVER DEGENERATE OCTAVES

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Abstract

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MATHEMATICS

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GEOMETRIES OVER DEGENERATE OCTAVES

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1. In the present note the projective plane $P_2(i, j, \varepsilon)$ over the algebra of degenerate octaves $R(i, j, \varepsilon)$ is considered, and the connection is indicated between the projective and non-Euclidean geometries of this plane and the limiting special groups of types G_2, F_4, E_6. All considerations are carried out over the field of real numbers.

2. Let $\mathfrak{G}$ be a connected Lie group, G its Lie algebra, and

$$G = K + E \tag{1}$$

a Cartan decomposition of the Lie algebra G, i.e.

$$[K, K] \subset K; \qquad [E, E] \subset K; \qquad [K, E] \subset E.$$

Following F. A. Berezin and I. M. Gelfand (([1]), p. 349), we define the limiting Lie group $\mathfrak{G}_0$ for the group $\mathfrak{G}$ with respect to the Cartan decomposition (1) as follows. Let $\mathfrak{K}$ be the connected subgroup of the group $\mathfrak{G}$ corresponding to the Lie subalgebra K. If $g \in \mathfrak{K}$, then $A_{gE} \subset E$, where A_g is the image of the element g under the adjoint representation.

The group $\mathfrak{G}_0$ is constructed on the direct product of the group $\mathfrak{K}$ and the vector space E. Namely:

$$(k_1, e_1)(k_2, e_2) \overset{\text{def}}{=} (k_1 k_2,\, A_{k_2}^{-1} e_1 + e_2),$$

where $k_i \in \mathfrak{K}$, $e_i \in E$ $(i = 1, 2)$.

In addition to the group $\mathfrak{G}_0$, we define the limiting Lie group with similitudes $\overline{\mathfrak{G}}_0$ for the group $\mathfrak{G}$ with respect to the same Cartan decomposition (1). We construct the group $\overline{\mathfrak{G}}_0$ on the direct product of the group $\mathfrak{G}_0$ and the set R of positive real numbers, setting

$$(k_1, e_1, \lambda_1)(k_2, e_2, \lambda_2) \stackrel{\text{def}}{=} (k_1, k_2,\ \lambda_2^{-1} A_{k_2}^{-1} e_1 + e_2,\ \lambda_1\lambda_2),$$

where $k_i \in \mathfrak{K}$, $e_i \in E$, $\lambda_i \in R$ $(i = 1, 2)$.

It is easy to verify the correctness of the definition introduced, and also the fact that the naturally embedded space E forms in $\mathfrak{G}_0$ (and in $\overline{\mathfrak{G}}_0$) a commutative normal divisor.

3. We now consider the algebra of degenerate octaves $R(i, j, \varepsilon)$. The linear space of the algebra $R(i, j, \varepsilon)$ is the direct sum of two linear spaces of the quaternion algebra $R(i, j)$, so that $\dim R(i, j, \varepsilon) = 8$. Multiplication in $R(i, j, \varepsilon)$ is defined by the formula

$$(x_1, y_1)(x_2, y_2) = (x_1x_2,\ y_2x_1 + y_1\bar{x}_2).$$

The linear mapping

$$J : R(i, j, \varepsilon) \to R(i, j, \varepsilon), \qquad (x, y) \mapsto \overline{(x, y)} = (\bar{x}, -y)$$

is an involution (i.e., an involutive antiautomorphism) in $R(i, j, \varepsilon)$.

The zero divisors (together with zero) have the form $(0, y)$ and form in $R(i, j, \varepsilon)$ an ideal E with trivial multiplication.

4. Let $\mathfrak{M}_3$ denote the space of square matrices of order 3 with entries from the algebra $R(i, j, \varepsilon)$. For $A = \|a_{ij}\| \in \mathfrak{M}_3$, put

$$A^* = \Gamma \widetilde{A}^t \Gamma^{-1},$$

where

$$\widetilde{A}^t = \|(\overline{a_{ij}})_{ji}\|, \qquad \Gamma = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3), \quad \gamma_i = \pm 1.$$

Put

$$\mathfrak{M}_3^+(\Gamma) = \{A :\ A \in \mathfrak{M}_3;\ A^* = A\}.$$

On the space $\mathfrak{M}_3^+(\Gamma)$ we introduce the structure of a Jordan algebra with identity $\mathfrak{B}(R(i, j, \varepsilon), \Gamma)$, putting

$$AB = \frac{1}{2}(A \cdot B + B \cdot A),$$

where $A, B \in \mathfrak{M}_3^+(\Gamma)$; $A \cdot B$ is the ordinary product of the matrices A and B. In what follows we shall consider only the cases

$$\Gamma = \Gamma_+ = \operatorname{diag}(1, 1, 1), \qquad \Gamma = \Gamma_- = \operatorname{diag}(1, 1, -1).$$

We also define

$$\mathfrak{B}(E,\Gamma)=\{A=\|a_{ij}\|:\ A\in\mathfrak{B}(R(i,j,\varepsilon),\Gamma);\ a_{ij}\in E\}.$$

Clearly, $\mathfrak{B}(E,\Gamma)$ is an ideal in $\mathfrak{B}(R(i,j,\varepsilon),\Gamma)$ with trivial multiplication.

5. To construct the projective plane, we introduce into consideration the spaces

$$\Pi_\Gamma=\{X:\ X\in\mathfrak{B}(R(i,j,\varepsilon),\Gamma)\setminus\mathfrak{B}(E,\Gamma);\ X^2=\operatorname{Sp}X\cdot X\},$$

where $\operatorname{Sp}X$ is the trace of the matrix X, and

$$\Pi^*_\Gamma=\{\rho X\},$$

where $X\in\Pi_\Gamma$, $\rho\neq 0$ is real; that is, Π^*_Γ is the set (or manifold) of lines passing through 0 and lying in Π_Γ.

We now define the projective plane $P_2(i,j,\varepsilon)$ over the algebra $R(i,j,\varepsilon)$ as the union of two sets—the set of points P and the set of lines L—with an incidence relation defined between the elements of these sets: $x\circ a$, where $x\in P$, $a\in L$, satisfying the following properties: there exist two one-to-one mappings

$$\mu_1:\ P\xrightarrow{\text{onto}}\Pi^*_{\Gamma_+},\qquad \mu_2:\ L\xrightarrow{\text{onto}}\Pi^*_{\Gamma_+}$$

such that, if $\rho X=\mu_1 x$, $\rho A=\mu_2 a$, then

$$x\circ a\Longleftrightarrow XA=0.$$

6. We now define the elliptic polar transformation (elliptic polarity) π_+:

$$\pi_+:\ P\to L,\qquad x\mapsto a=\mu_2^{-1}\mu_1 x.$$

Clearly,

$$x\circ a\Longleftrightarrow \pi_+^{-1}a\circ\pi_+x.$$

The projective plane $P_2(i,j,\varepsilon)$ with the given polarity π_+ will be called the elliptic unitary plane $\overline{S}_2(i,j,\varepsilon)$ over the algebra $R(i,j,\varepsilon)$ (cf. (2)).

Next, we define the hyperbolic unitary plane ${}^1\overline{S}_2(i,j,\varepsilon)$. For this purpose consider the one-to-one mappings

$$\psi_1:\ \Pi^*_{\Gamma_+}\to\Pi^*_{\Gamma_-},\qquad \rho\begin{vmatrix}\lambda_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \lambda_2 & x_1\\ x_2 & \bar{x}_1 & \lambda_3\end{vmatrix}\mapsto\rho\begin{vmatrix}\lambda_1 & x_3 & -\bar{x}_2\\ \bar{x}_3 & \lambda_2 & -x_1\\ x_2 & \bar{x}_1 & -\lambda_3\end{vmatrix};$$

$$\psi_2:\ \Pi^*_{\Gamma_+}\to\Pi^*_{\Gamma_-},\qquad \rho\begin{vmatrix}\lambda_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \lambda_2 & x_1\\ x_2 & \bar{x}_1 & \lambda_3\end{vmatrix}\mapsto\rho\begin{vmatrix}\lambda_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \lambda_2 & x_1\\ -x_2 & -\bar{x}_1 & -\lambda_3\end{vmatrix}.$$

The hyperbolic polar transformation $\pi_{\_}$ is defined as follows:

$$\pi_{\_}: \ P \to L, \qquad x \mapsto a = \mu_2^{-1}\psi_2^{-1}\psi_1\mu_1 x.$$

It turns out that

$$x \circ a \Longleftrightarrow \pi_{\_}^{-1} a \circ \pi_{\_} x.$$

By the plane ${}^1\bar{S}_2(i, j, \varepsilon)$ we shall mean the plane $P_2(i, j, \varepsilon)$ with the polarity $\pi_{\_}$ defined above.

In the spaces $\bar{S}_2(i, j, \varepsilon)$ and ${}^1\bar{S}_2(i, j, \varepsilon)$ one can introduce the structures of semi-Riemannian spaces (see, for example, ([2])), defining in them a metric as follows. Consider the mappings

$$p_1 : \bar{S}_2(i, j, \varepsilon) \xrightarrow{R_8} \bar{S}_2(i, j), \qquad p_2 : {}^1\bar{S}_2(i, j, \varepsilon) \xrightarrow{{}^4R_8} {}^1\bar{S}_2(i, j)$$

(see ([2,3])).

Here by unitary non-Euclidean planes are meant the sets of interior points, R_8 is eight-dimensional Euclidean space; ${}^4R_8$ is a pseudo-Euclidean space of index 4. The mappings p_i $(i = 1, 2)$ are fibrations. We shall denote the distance between points x and y by $d(x, y)$. Then

$$d(x, y) \overset{\text{def}}{=} d(p_i x, p_i y), \qquad \text{if} \quad p_i x \neq p_i y,$$

$$d(x, y) \overset{\text{def}}{=} d_R(x, y), \qquad \text{if} \quad p_i x = p_i y,$$

where $d_R(x, y)$ is the distance in the metric of the fiber.

7. A collineation φ of the plane $P_2(i, j, \varepsilon)$ will mean a pair of one-to-one mappings $\varphi = (\varphi_1, \varphi_2)$

$$\varphi_1 : P \overset{\text{onto}}{\longrightarrow} P, \qquad \varphi_2 : L \overset{\text{onto}}{\longrightarrow} L$$

such that

$$x \circ a \Longleftrightarrow \varphi_1 x \circ \varphi_2 a.$$

In what follows we shall write $\varphi x = \varphi_1 x$, $\varphi a = \varphi_2 a$.

A similarity of the plane $\bar{S}_2(i, j, \varepsilon)$ (or ${}^1\bar{S}_2(i, j, \varepsilon)$) is a collineation φ such that $\varphi\pi_+ = \pi_+\varphi$ (respectively, $\varphi\pi_{\_} = \pi_{\_}\varphi$). A motion of the planes $\bar{S}_2(i, j, \varepsilon)$, ${}^1\bar{S}_2(i, j, \varepsilon)$ is a similarity preserving the structure of semi-Riemannian spaces, i.e. the metric $d(x, y)$.

8. **Theorem 1.** The group of automorphisms of the algebra $R(i,j,\varepsilon)$ is isomorphic to the group $(\bar{G}_2^+)_0$, where G_2^+ is the compact group of type G_2.

Theorem 2. The group of collineations of the plane $P_2(i,j,\varepsilon)$ is isomorphic to the group $(\bar{E}_6^-)_0$, where E_6^- is the noncompact group of type E_6, which is the group of collineations of the octavian projective plane $P_2(i,j,k)$ (see, for example, ([3])).

Theorem 3. The groups of similarities and motions of the plane $\bar{S}_2(i,j,\varepsilon)$ are respectively isomorphic to the groups $(\bar{F}_4^+)_0$ and $(F_4^+)_0$, where F_4^+ is the compact group of type F_4.

Theorem 4. The groups of similarities and motions of the plane ${}^1\bar{S}_2(i,j,\varepsilon)$ are respectively isomorphic to the groups $(\bar{F}_4^-)_0$ and $(F_4^-)_0$, where F_4^- is the noncompact group of type F_4, which is the group of motions of the octavian unitary hyperbolic plane ${}^1S_2(i,j,k)$.

9. A line $P_1(i,j,\varepsilon)$ of the plane $P_2(i,j,\varepsilon)$, considered as the set of points incident with one and the same line, is a cylinder $S^4 \times R^4$ with a 4-dimensional sphere S^4 as base and a 4-dimensional plane R^4 as generator. The line $P_1(i,j,\varepsilon)$ may be regarded as the absolute in the semi-Euclidean space ${}^1S_9^5$ (see ([4])). The group of motions of the space ${}^1S_9^5$ induces on the absolute the group C of conformal transformations, and the group of similarities induces the group $\bar{C}$ of conformal transformations with similarity.

On the other hand, the line $P_1(i,j,\varepsilon)$, considered as a submanifold of the semi-Riemannian space $\bar{S}_2(i,j,\varepsilon)$ (or ${}^1\bar{S}_2(i,j,\varepsilon)$), is isometric to a connected component of the sphere of the semi-Euclidean space R_9^5 (respect-

respectively, ${}^{1,0}R_9^5$). Therefore, after identifying diametrically opposite points, the lines of the planes $\bar{S}_2(i,j,\varepsilon)$, ${}^1\bar{S}_2(i,j,\varepsilon)$ will be isometric respectively to the spaces S_8^4 and ${}^{1,0}S_8^4$ (see (4)). Moreover, the following theorems hold:

Theorem 5. *The group of collineations $(\bar{E}_6^-)_0$ of the plane $P_2(i,j,\varepsilon)$ induces on the line $P_1(i,j,\varepsilon)$ the group $\bar{C}$, while the subgroup $(E_6^-)_0$ induces the group C (at least locally).*

Theorem 6. *The group of similarities of the plane $\bar{S}_2(i,j,\varepsilon)$ (respectively, ${}^1S_2(i,j,\varepsilon)$) induces on the line, as on the space S_8^4 (respectively, on the space ${}^{1,0}S_8^4$), the group of similarities, and the group of motions—the group of motions (at least locally).*

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Note: Figure translations are in progress. See original paper for figures.