ON THE METRIC THEORY OF LINEAR DIOPHANTINE APPROXIMATIONS

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Abstract

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ON THE METRIC THEORY OF LINEAR DIOPHANTINE APPROXIMATIONS

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By virtue of a well-known theorem of A. Ya. Khinchin ([1,3,7]), the inequalities

$$\|\omega q\| < \varphi(q) \qquad (q = 1, 2, \dots)$$

for almost all real numbers ω (“almost all” in the sense of Lebesgue measure) have only a finite number of solutions if the series

$$\sum_{q=1}^{\infty} \varphi(q), \tag{1}$$

converges, and an infinite number of solutions if the numbers $\varphi(q)$ decrease monotonically and the series (1) diverges. Here and below $\|\alpha\|$ denotes the distance from α to the nearest integer. Duffin and Schaeffer ([4]) observed that in the latter assertion one cannot drop the condition of monotone decrease of the numbers $\varphi(q)$. Cassels ([1,5]) extended Khinchin’ s result to simultaneous approximations, considering the occurrence of the points $(\|\omega_1 q\|, \dots, \|\omega_m q\|)$ in rectangular domains, and Gallagher ([8]) generalized Cassels’ result to hyperbolic domains (in fact his arguments carry over to the case of star-shaped domains), and showed that in the case of rectangular domains the monotonicity condition can be dropped if $m \geq 2$ ([6]). W. Schmidt ([9]) considered the case of simultaneous homogeneous approximations with several linear forms and obtained an analogue of Khinchin’s theorem without assumptions analogous to the condition of monotone decrease of the numbers $\varphi(q)$ in Khinchin’ s theorem.

In this note we show that Khinchin’ s theorem extends to the case of simultaneous inhomogeneous approximations with linear forms in two or more variables (I) and, more generally, that analogous results carry over to the geometry of numbers (II).

I. Let R^n be n-dimensional Euclidean space, E_n the unit cube in R^n, and Ω_{mn} the space of $m \times n$ matrices (ω_{ij}) over E_1, i.e., the set of matrices

$$\omega = \begin{pmatrix} \omega_{11}\ \omega_{12}\ \dots\ \omega_{1n} \\ \dots\dots\dots \\ \omega_{m1}\omega_{m2}\dots\omega_{mn} \end{pmatrix} \tag{2}$$

with the condition $0 \le \omega_{ij} < 1$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$. We assume that on the space Ω_{mn} a Lebesgue measure is defined, induced by a one-to-one mapping of the space Ω_{mn} onto the unit mn-dimensional cube E_{mn}. Let A_{mn} denote the class of measurable sets over Ω_{mn}. For an integral vector $a \in R^n$, $a \ne (0)$, define a mapping $T = T(a)$ of the space Ω_{mn} into Ω_{1m} by

$$T:\ \omega \to (\{a\omega_1\}, \dots, \{a\omega_m\}), \tag{3}$$

where $\{x\}$ is the fractional part of the number x, ω_i are the row vectors of matrix (2), and $a\alpha$ is the scalar product of the vectors a and α. If $A \in \mathbf{A}_{1m}$, then let $T^{-1}A$ denote the complete preimage of A in Ω_{mn} under the mapping (3). The mappings (3) preserve measure, i.e.,

$$|T^{-1}A| = |T|, \tag{4}$$

and for linearly independent vectors $a_1, a_2 \in R_n$ the sets $T^{-1}A$ are stochastically independent: if $A_1, A_2 \in A_{mn}$, $T_1 = T(a_1)$, $T_2 = T(a_2)$, then

$$|T_1^{-1}A_1 \cap T_2^{-1}A_2| = |A_1| \cdot |A_2|. \tag{5}$$

These remarks are the starting point for the proof of the following assertions.

Theorem 1. Let $n \ge 2$, and let to each integral primitive vector $a \in R_n$ there be assigned a measurable set $A(a)$ lying in the m-dimensional unit cube E_m. Then the conditions

$$(\{a\omega_1\}, \dots, \{a\omega_m\}) \in A(a), \tag{6}$$

where $\omega_1, \dots, \omega_m$ are the row vectors of matrix (2), are satisfied for almost all matrices ω only finitely many times by integral primitive vectors a, if

$$\sum_a |A(a)| < \infty, \tag{7}$$

and are satisfied infinitely often by such vectors, if

$$\sum_a |A(a)| = \infty. \tag{8}$$

Theorem 2. Suppose that, under the conditions of the preceding theorem, $N(\omega, h)$ denotes the number of primitive vectors $a = (a_1, \dots, a_n)$ with the bound $h(a) = \max(|a_1|, \dots, |a_n|) \le h$, for which (6) is satisfied for a fixed matrix ω. Then for almost all ω

$$N(\omega, h) = \Phi(h) + O(\Phi^{1/2+\varepsilon}(h) \ln h),$$

where

$$\Phi(h) = \sum_{h(a)\le h} |A(a),$$

the summation being over all primitive n-dimensional vectors a with bound $h(a) \le h$.

The proof of Theorem 1 is obtained as follows. If (Ω, A, μ) is a measure space and an infinite sequence of sets $A_i \in A$ $(i = 1, 2, \ldots)$ is given, then in the case $\sum \mu A_i < \infty$ the set A of points which belong to infinitely many of the sets A_i has measure zero (Borel-Cantelli); whereas if $\sum \mu A_i = \infty$, then A has measure

$$\mu A \ge \overline{\lim} \frac{\left(\sum_{i\le n} \mu A_i\right)^2}{\sum_{k,l\le n} \mu(A_k \cap A_l)}. \tag{9}$$

The assertion of Theorem 1 in the case (7) is obtained if, as the sets A_i, one takes the sets $T_a^{-1}A(a)$, numbered according to increasing value of $h(a)$, and applies (4). In the case (8), note that if the vectors a_1, a_2 are primitive, $a_1 \ne \pm a_2$, then they are linearly independent. From (5) it follows that

$$\sum_{h(a),\, h(b)\le h} \left|T_a^{-1}A(a) \cap T_b^{-1}A(b)\right| \sim \left(\sum_{h(a)\le h} |A(a)|\right)^2$$

(the summation is over primitive vectors), so that, by virtue of (9), the second assertion of Theorem 1 is obtained.

Theorem 2 is derived from (5) by applying the properties of orthogonal random variables (we consider the random variables $\xi_a = I_a - |A(a)|$,

where I_a is the indicator of the set $T_a^{-1}A(a)$, a is a primitive vector, and we apply Theorem B. (11) from (2), p. 480). An analogue of this theorem for rectangular sets was obtained by W. Schmidt by the dispersion method ((9); see also (10)).

II. Let Λ_0 be the lattice of m-dimensional integer vectors; $\Gamma = \Gamma(\omega)$ the additive group generated by the column vectors $\omega^1, \omega^2, \ldots, \omega^n$ of the matrix (2), i.e., Γ is the set of all m-dimensional vectors γ representable in the form $\gamma = a_1\omega^1 + a_2\omega^2 + \cdots + a_n\omega^n$, with integral rational $a_1, a_2, \ldots, a_n$. We shall call the vector γ primitive if the vector $a = (a_1, a_2, \ldots, a_n)$ is primitive. It is clear that in Theorems 1 and 2 the question concerns the incidence of primitive points $\gamma \in \Gamma$ in the set $A(a)$ of the factor space R_m/Λ_0: condition (6) is equivalent to the condition

$$\gamma = a_1\omega^1 + \cdots + a_n\omega^n \in A(a) \pmod{\Lambda_0}. \tag{10}$$

The following two theorems generalize the preceding results to the case of replacing the lattice Λ_0 by an arbitrary nondegenerate lattice Λ.

Theorem 3. Let Λ be an arbitrary nondegenerate m-dimensional lattice, $n \geq 2$, and let to each integral primitive vector $a \in R_n$ there correspond a measurable set $A(a) = A_\Lambda(a)$ of the factor space R_m/Λ. Then the conditions

$$a_1\omega^1 + \cdots + a_n\omega^n \in A(a) \pmod{\Lambda}, \tag{11}$$

where $\omega^1, \ldots, \omega^n$ are the column vectors of the matrix (2), for almost all matrices ω are satisfied only a finite number of times by integral primitive vectors a, if (7) is true, and are satisfied infinitely often by such vectors, if (8) is true.

Theorem 4. Let, under the conditions of the preceding theorem, $N(\omega, h)$ denote the number of primitive vectors $a = (a_1, \ldots, a_n)$ with restriction $h(a) \leq h$, for which (11) is satisfied for a fixed matrix ω. Then for almost all ω

$$N(\omega, h) = \frac{1}{d(\Lambda)}\Phi(h) + O\left(\Phi^{1/2+\varepsilon}(h)\ln h\right),$$

where $d(\Lambda)$ is the determinant of the lattice Λ, $\Phi(h) = \sum\limits_{h(a)\leq h} |A(a)|$, and the summation is over all primitive n-dimensional vectors a with restriction $h(a) \leq h$.

Both last theorems can be derived from Theorems 1 and 2 by transforming condition (10) into (11) through multiplication by a matrix that carries the lattice Λ_0 into Λ. A direct proof using analogues of equalities (4), (5) is also possible.

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Note: Figure translations are in progress. See original paper for figures.