REGULARIZATION OF ILL-POSED PROBLEMS FOR SINGULAR INTEGRAL EQUATIONS

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Abstract

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REGULARIZATION OF ILL-POSED PROBLEMS FOR SINGULAR INTEGRAL EQUATIONS

(Presented by Academician A. N. Tikhonov, 19 XII 1966)

Numerous works on singular integral equations have been devoted to the construction of a regularizer that would reduce a singular integral equation to an equivalent Fredholm equation ([1–7]). At the same time, for a general closed linear operator A in Hilbert space, a necessary condition for the existence of a regularizer is the closedness of the range $R(A)$ ([5,7]). This is equivalent to the well-posedness of the problem $Au = f$, provided that u is orthogonal to the subspace of solutions of $Av = 0$.

Therefore, for an ill-posed integral equation with a singular kernel we shall pose a different regularization problem–regularization in the sense of A. N. Tikhonov ([8,9]), i.e., we shall abandon the principal requirement: the equivalence of the regularized and the original problems. The discussion will concern only the uniform convergence of solutions of a family of regularized equations to some solution of the original problem. We shall prove that a certain modification of A. N. Tikhonov' s regularization, developed by him for bounded operators, is also applicable to unbounded operators, i.e., to integral equations with singular kernels, and also to ill-posed linear integro-differential equations with partial derivatives.

Let H be a Hilbert space, and let T be a closed operator in H with dense domain $D(T)$. Consider the equation

$$Tv = f, \tag{1}$$

where $v \in D(T)$, $f \in R(T)$ is given.

In the case of bounded operators, the regularization proposed for this problem by A. N. Tikhonov has the form

$$(T^*T + \delta)x_\delta = T^*f_\delta; \qquad \delta \leq \|f_\delta - f\|. \tag{2}$$

We shall suppose that the operator T is unbounded. Assume first that $f_\delta \in D(T^*)$.

Denote by P_T, P_{T^*} the projection operators onto the subspaces of solutions of $Tu = 0$ and $T^*u = 0$, respectively; by T_1 the restriction of the operator T to the domain* $(1 - P_T)D(T)$; and by T_{1*} the restriction of the operator T^* to the domain $(1 - P_{T^*})D(T^*)$. It is obvious that: 1) T_1^{-1}, T_{1*}^{-1} exist; 2) $TT_1^{-1} = 1$, $\overline{T_1^{-1}T} = 1 - P_T$; 3) $\overline{P_{TT}1^{-1}} = 0$; $\overline{P_T^* T_{1*}^{-1}} = 0$; 4) $\overline{P_{T^*}T} = 0$, $\overline{P_{TT}^*} = 0$. (The bar denotes closure.) The operator $A = T_1^{-1}(1 - P_{T^*})$ is defined on an everywhere dense set in H, and therefore the operators A^* and A^*A, whose domains are dense, exist. It is not difficult to verify that $A^* = T_{1*}^{-1}(1 - P_T)$. We shall prove this assertion.

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* By P_{TH} we mean the subspace of solutions of $Tx = 0$; $(1 - P_T)H$ is its orthogonal complement. By $(1 - P_T)D$, where $D \subset H$, we mean the domain equal to $D \cap (1 - P_T)H$.

Proof. Let $f \in D[T_{1*}^{-1}(1 - P_T)] = R(T^*) \oplus P_T H$, and let $F = T_{1*}^{-1}(1 - P_T)f$; then $P_{T_*}F = 0$ and $(1 - P_T)f = T^*F$, while

$$(f, Ag) = (f, T_1^{-1}(1 - P_{T*})g) = (f(1 - P_T)T_1^{-1}(1 - P_{T*})g) = ([1 - P_T]f, T_1^{-1}[1 - P_{T*}]g)$$

$$= (T^*F, T_1^{-1}[1 - P_{T*}]g) = (F, [1 - P_{T*}]g) = (F, g),$$

i.e. $A^*f = F \Rightarrow T_{1*}^{-1}(1 - P_T) \subset A^*$. Let $g \in D(T)$, $h \in D(A^*)$, $A^*h = f$; then

$$(f, Tg) = (h, ATg) = ([1 - P_T]h, g) \Rightarrow T^*f = (1 - P_T)h \Rightarrow (1 - P_T)D(A^*) = R(T^*) \Rightarrow D(A^*) = R(T^*) \oplus P_T H$$

$$\Rightarrow A^* = T_{1*}^{-1}(1 - P_T).$$

As indicated above, the operator

$$AA^* = T_1^{-1}(1 - P_{T*})T_{1*}^{-1}(1 - P_T) = T_1^{-1}T_{1*}^{-1}(1 - P_T)$$

has an everywhere dense domain of definition. Consequently, the operator $T_1^{-1}T_{1*}^{-1}$ is defined on a domain everywhere dense in $(1 - P_T)H$.

By the known von Neumann lemma, for any closed operator B with an everywhere dense domain of definition, the operators $(1 + B^*B)^{-1}$ and $B(1 + B^*B)^{-1}$ are defined on all of H and are bounded by one. Consequently, $(B[1 + B^*B])^* = (1 + B^*B)B^*$ is bounded by one.

Putting $B = \delta^{-1/2}T$, we arrive at the conclusion that the operators

$$C_\delta = (\delta + T^*T)^{-1}$$

and

$$B_\delta = (\delta + T^*T)^{-1}T^*$$

are bounded respectively by δ^{-1} and $\delta^{-1/2}$.

It is also obvious that the operator

$$R_\delta = (\delta + T^*T)^{-1}T^*T_1$$

is defined on $(1 - P_T)H$ and is bounded by two. Indeed,

$$\|R_\delta\| = \|(\delta + T^*T)^{-1}T^*T_1\| \le \|(\delta + T^*T)^{-1}T^*T\|$$
$$= \|1 - \delta(\delta + T^*T)^{-1}\| \le 2.$$

Since $P_{T*}T_1 = P_{T*}T = 0$, and hence $T^*T_1 = T_{1*}T_1$, we have

$$R_\delta = (\delta + T^*T)^{-1}T_{1*}T_1.$$

It follows from this that

$$R_\delta^{-1} = T_1^{-1}T_{1*}^{-1}(\delta + T^*T) = \delta T_1^{-1}T_{1*}^{-1} + 1 - P_T$$

exists and is defined on a domain dense in $(1 - P_T)H$. On elements of this domain, obviously, $R_\delta^{-1} \to 1$ as $\delta \to 0$.

Let $z = (1 - P_T)v$.

Since $\|R_\delta\| \le 2$, by Theorem 3.2 of Part I of the book [10], we obtain that R_δ converge as $\delta \to 0$ to 1 on the subspace $(1 - P_T)H$. Consequently,

$$x_\delta - z = B_\delta f_\delta - z = B_\delta(f_\delta - f) + B_\delta f - z$$
$$= B_\delta(f_\delta - f) + B_\delta T_1 z - z = B_\delta(f_\delta - f) + R_\delta z - z \to 0 \quad \text{as } \delta \to 0,$$

since

$$\|B_\delta(f_\delta - f)\| \le \frac{1}{\sqrt{\delta}}\|f_\delta - f\| = \sqrt{\delta} \to 0,$$

and $z \in (1 - P_T)H$, and hence $\|R_\delta z - z\| \to 0$ as $\delta \to 0$.

In solving the well-posed problem (2), we proceeded from the fact that $f_\delta \in D(T^*)$. This restriction can be avoided by means of the following procedure.

Consider the equation

$$(T^*T + \delta_1)y_\delta = T^*u_\delta, \qquad \delta_1 = \|[(1 + \delta TT^*)^{-1} - 1]f_\delta\|, \tag{3}$$

where u_δ, in turn, satisfies the equation

$$(1 + \delta TT^*)u_\delta = f_\delta, \qquad f_\delta \in H; \qquad \|f_\delta - f\| \le \delta, \quad f \in R(T), \tag{4}$$

f_δ, f are given.

In this case it is obvious that T^*u_δ exists for any $f_\delta \in H$. Moreover, $\delta_1 \to 0$ as $\delta \to 0$, and

$$\|u_\delta - f\| \le \delta + \delta_1 \to 0 \quad \text{as } \delta \to 0.$$

Indeed,

$$\delta_1 = \|[(1+\delta TT^*)^{-1}-1](f_\delta - f + f)\| \le 2\delta + \|[(1+\delta TT^*)^{-1}-1]f\| \to 0 \quad \text{as } \delta \to 0,$$

and

$$\|u_\delta - f\| = \|(1+\delta TT^*)^{-1} f_\delta - f_\delta + (f_\delta - f)\| \le \|[(1+\delta TT^*)^{-1}-1]f_\delta\| + \delta = \delta_1 + \delta.$$

Thus, in view of the preceding considerations, $y_\delta - z \to 0$ as $\delta \to 0$.

Suppose now that $H = L_2[R^n]$. Consider

$$v_{\delta_1,\varepsilon} = \varepsilon^{-(n+2)/2}(R_{\delta_1} - 1)(1 - P_T)e^{-(x-\xi)^2/4\varepsilon}, \qquad x, \xi \in R^n,$$

i.e., $v_{\delta_1,\varepsilon}$ satisfies the equation

$$(\delta_1 + T^*T)v_{\delta_1,\varepsilon} = \delta_1(1 - P_T)e^{-(x-\xi)^2/4\varepsilon}\varepsilon^{-(n+2)/2}.$$

By what has been proved, for fixed ε the function $v_{\delta_1,\varepsilon}$ tends to zero in norm as $\delta_1 \to 0$. Consequently, there exists a function $\varepsilon = \varepsilon(\delta_1) \to 0$ as $\delta_1 \to 0$ such that $\varepsilon(\delta_1) > \delta_1^{2/(4+n)}$ and $\|v_{\delta,\varepsilon(\delta_1)}\|$ is bounded above as $\delta_1 \to 0$.

Let us prove that if $z(x) \in (1 - P_T)L_2(R^n) \cap C^2$ and $Tz(x) = f$, then

$$\max_x \left| z(x) - \frac{1}{(2\pi\varepsilon(\delta_1))^{n/2}} \int y_\delta(\xi) e^{-(x-\xi)^2/4\varepsilon(\delta_1)}\, d\xi \right| = O[\varepsilon(\delta_1)]_{\delta \to 0}. \tag{5}$$

Indeed,

$$\begin{aligned}
&z(x) - \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon} y_\delta(\xi)\, d\xi \\
&\quad = \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon}[z(x) - z(\xi)]\, d\xi + \frac{1}{(2\pi\varepsilon)^{n/2}} \int e^{-(x-\xi)^2/4\varepsilon}[z(\xi) - y_\delta(\xi)]\, d\xi \\
&\quad \le \varepsilon \max z_{x_i x_j} + \left| \frac{1}{(2\pi\varepsilon)^{n/2}} \left(B_{\delta_1}[y_\delta - f] + B_{\delta_1} f - z,\ e^{-(x-\xi)^2/4\varepsilon} \right) \right| \\
&\quad \le \varepsilon \max z_{x_i x_j} + \frac{\delta_1 + \delta}{\sqrt{\delta_1}(2\pi\varepsilon)^{n/2}} \left\| e^{-(x-\xi)^2/4\varepsilon} \right\| + \frac{1}{(2\pi\varepsilon)^{n/2}} \left| \left((R_{\delta_1} - 1)(1 - P_T)z,\ e^{-(x-\xi)^2/4\varepsilon} \right) \right| \\
&\quad \le \varepsilon \max z_{x_i x_j} + \frac{(\delta_1 + \delta)(2\pi\varepsilon)^{-n/4}}{\sqrt{\delta_1}} + \|z\|\, \varepsilon\, \|v_{\delta_1,\varepsilon}\| = O(\varepsilon).
\end{aligned}$$

Thus we have proved the following theorem:

Theorem. *For any closed linear operator T with everywhere dense domain of definition in $L_2[R^n]$, problem (3)–(4) is well posed, and its solution y_δ satisfies* *relation (5), if there exists a solution of the equation $Tz(x) = f$ belonging to $C^q \cap (1 - P_T)D(T)$.*

We note that under the additional condition of Tikhonov well-posedness of problem (1), using the effective technique of M. M. Lavrent' ev (11), one can obtain refined estimates for the regularization presented here.

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Note: Figure translations are in progress. See original paper for figures.