On a Method for Organizing Sliding in a Dynamic System

Authors: N. G. Yarovoi

Date: 1967-01-01T00:00:00+00:00

Abstract

The paper considers a third-order dynamical system with a variable structure. The switching surface is defined by a pair of intersecting planes. Necessary and sufficient conditions for the existence of a sliding mode, stability conditions for motion along the switching surface, and necessary and sufficient conditions for the representative point to reach the sliding surface are established. Bibliography: 7 items.

Full Text

Preamble

The study of variable structure systems has been extensively developed in the works of N. G. Barbashin and others [?, ?, ?, ?, ?]. In particular, it has been shown that for a system of the form

$$\dot{x} = z, \quad \dot{z} = -bz - az - ab,$$

where a and b are parameters and the control law is defined by $s = \text{sign}\ x(Ax + By + z)$, a sliding mode can be established on the switching surface. In this context, we consider a third-order system of the following form:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ab, \end{aligned} \qquad (1.1)$$

where $i = 1, 2, 3$ and the coefficients $|a_{ik}| < 1$. The control function is defined as $a = \text{sign}\ (Ax_1 + Bx_2 + Cx_3)(Ax_1 + Bx_2 - Cx_3)$, where A, B, C are constants. The switching surfaces are given by $Ax_1 + Bx_2 + Cx_3 = 0$ and $Ax_1 + Bx_2 - Cx_3 = 0$.

1. Stability Conditions and Sliding Modes

To ensure the existence of a sliding mode on the intersection of the switching surfaces, we examine the derivative of the Lyapunov-like function $v = (Ax_1 + Bx_2 + Cx_3)(Ax_1 + Bx_2 - Cx_3)$. The condition for the existence of a sliding mode [?] requires that $\lim_{s\to 0} \frac{dv}{dt} < 0$. For the system (1.1), this leads to the following requirements for the coefficients:

$$\begin{aligned} D &= A + a_{21}B - a_{23} - A - a_{12}A + a_{22}B + a_{32}C - a_{33}B - ab, \\ D_1 &= A + a_{21}B - a_{31}C + a_{13} - a_{33}A - a + a_{12}A + a_{22}B - a_{32}C + a_{33}B - ab. \end{aligned} \tag{1.2}$$

The sliding mode exists if $D < 0$ and $D_1 < 0$ when $a = +1$, and $D > 0, D_1 > 0$ when $a = -1$. Under these conditions, the system trajectories are constrained to the switching manifold.

Further analysis of the sliding equations (1.3) shows that if $AE - BD = 0$ and $AE_1 - BD_1 = 0$, the system simplifies significantly. Specifically, if we assume $a_{31} = 0$ and $a_{22} + A^2a_{12} - ab - a_{21}B^2 = 0$, the sliding motion is governed by:

$$\begin{aligned} \dot{x}_1 &= 2D(Ax_1 + Bx_2)^2, \\ \dot{x}_2 &= 2D_1(Ax_1 + Bx_2)^2. \end{aligned} \tag{1.4}$$

For stability, we require $D < 0$ and $D_1 < 0$ when $a = +1$.

2. Analysis of the Switching Manifold

We consider the behavior of the system near the intersection of the planes $Ax_1 + Bx_2 + Cx_3 = 0$ and $Ax_1 + Bx_2 - Cx_3 = 0$. Following the methodology in [?], the dynamics on these planes can be described by the reduced system:

$$\begin{aligned} \dot{x}_1 &= (a_{11} + a_{13}\frac{A}{C})x_1 + (a_{12} + a_{13}\frac{B}{C})x_2, \\ \dot{x}_2 &= (a_{21} + a_{23}\frac{A}{C})x_1 + (a_{22} + a_{23}\frac{B}{C})x_2. \end{aligned} \tag{2.1}$$

The stability of the equilibrium point $(0,0,0)$ depends on the eigenvalues of the matrix associated with (2.1) and (2.2). The conditions for the trajectories to reach the switching manifold from the surrounding phase space are given by the inequalities:

$$|Aa_{13} + Ba_{23}| < -C(a_{11} - a_{21}), \tag{2.3}$$

which ensure that the sliding mode is attractive.

3. Global Dynamics and Convergence

To analyze the global behavior of system (1.1), we introduce a coordinate transformation $z_1 = a_{11}x_1 + b_{13}z_3$ and $z_2 = x_2$. In the new coordinate system

(z_1, z_2, z_3), the equations of motion become:

$$\begin{aligned} \dot{z}_1 &= b_{11}z_1 + b_{13}z_3, \\ \dot{z}_2 &= b_{21}z_1 + b_{22}z_2 + b_{23}z_3, \\ \dot{z}_3 &= b_{31}z_1 + b_{32}z_2 + b_{33}z_3 + ab. \end{aligned} \quad (3.2)$$

We define the region of interest where $z_3 > 0$ and $z_3 - 2Az_1 < 0$. In this region, the control $a = +1$ is active. The characteristic equation for the linear part of the system is:

$$\lambda^2 - (b_{11} + b_{33} + b)\lambda + b_{13}(b_{31} - Ab) = 0. \quad (3.4)$$

Let λ_2 and λ_3 be the roots of this equation. If $\lambda_3 - \lambda_2 > 0$, the system exhibits stable convergence toward the sliding manifold. For an initial point $M_0(z_1^0, z_2^0, z_3^0)$ satisfying $z_3^0 > 0$, the time t_1 required to reach the switching surface $z_3 - 2Az_1 = 0$ is determined by:

$$t_1 = \frac{1}{\lambda_3 - \lambda_2} \ln \left[\frac{2A(\lambda_2 - b_{33} - b) - b_{31} + bA}{2A(\lambda_3 - b_{33} - b) - b_{31} + bA} \right]. \quad (3.6)$$

The condition for reaching the surface in finite time is $t_1 > 0$, which imposes constraints on the system parameters (3.7) and (3.8). Specifically, we require $2A(\lambda_2 - b_{33}) - A < 0$ to ensure that the trajectories do not diverge before hitting the switching plane.

Similar analysis applies to the region where $a = -1$. By combining the conditions for both control states, we establish the global asymptotic stability of the system under the proposed variable structure control law.

References

1. Barbashin, E. A., Tabueva, V. A., *Automation and Remote Control*, No. 3, 183-188, 1962.
2. Barbashin, E. A., Tabueva, V. A., *Automation and Remote Control*, 23, No. 10, 1962.
3. Erugin, N. P., *Applied Mathematics and Mechanics*, 152, No. 2, 299-301, 1963.
4. Gelig, A. Kh., *Automation and Remote Control*, No. 4, 157-163, 1963.
5. Drozdov, Yu. V., *Proceedings of the 1st IFAC Congress*, 1955, pp. 421-438.
6. Alimov, Yu. I., *Proceedings of the 2nd IFAC Congress*, 1961, pp. 679-688.
7. Utkin, V. I., *Automation and Remote Control*, No. 3, 132-138, 1965.

Note: Figure translations are in progress. See original paper for figures.