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Abstract

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V. A. KRASNIKOV

ON THE QUESTION OF THE HYDRODYNAMIC APPROXIMATION OF THE GREEN' S FUNCTIONS OF A SUPERFLUID BOSE SYSTEM

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At present there is no general method for decoupling the chain of equations for Green' s functions. In this connection, the asymptotic calculation (for small k and ω) of Green' s functions that are directly connected with the physical characteristics of the system, and that requires no approximations, is of interest. In the present note, following the ideas of N. N. Bogolyubov ([1], [2]), the Green' s functions are calculated in the viscous hydrodynamic approximation. Let us note that this question was also studied in ([3]). However, in contrast to ([3]), here additional external fields are included in the Hamiltonian of the system, making it possible to obtain new Green' s functions, and the restriction of weak dissipation is not imposed. In ([4]) the asymptotic form of the correlation functions for superfluid systems was obtained by another method, which, however, does not appear to us to be simpler. The case of normal systems was studied in ([5], [6]).

As is known ([1]), for a superfluid system there is a statistical expression, characteristically associated with the presence of a separated condensate, for the removal of which terms with particle "sources" $\eta(t, \mathbf{r})$ should be introduced into the Hamiltonian. Therefore let us take the Hamiltonian in the form

$$\begin{aligned} H = \frac{1}{2m} \int \nabla\Psi^+(t,\mathbf{r})\,\nabla\Psi(t,\mathbf{r})\,d\mathbf{r} + \\ + \frac{1}{2}\int \Phi(|\mathbf{r}-\mathbf{r}'|)\Psi^+(t,\mathbf{r})\Psi^+(t,\mathbf{r}')\Psi(t,\mathbf{r}')\Psi(t,\mathbf{r})\,d\mathbf{r}\,d\mathbf{r}' + \\ + \int (U(t,\mathbf{r})-\lambda)\Psi^+(t,\mathbf{r})\Psi(t,\mathbf{r})\,d\mathbf{r} - \frac{1}{m}\int \mathbf{j}(t,\mathbf{r})\mathbf{A}(t,\mathbf{r})\,d\mathbf{r} + \\ + \int \{\eta(t,\mathbf{r})\Psi^+(t,\mathbf{r}) + \eta^*(t,\mathbf{r})\Psi(t,\mathbf{r})\}\,d\mathbf{r}, \end{aligned} \tag{1}$$

where λ is the chemical potential, and $U(t, \mathbf{r})$ and $\mathbf{A}(t, \mathbf{r})$ are the potential and vortical components of a certain external field satisfying the condition div $\mathbf{A} = 0$.

The introduction into the Hamiltonian of terms with external "sources" U, $\mathbf{A}$, and η is required in order to take variations of the hydrodynamic averages with respect to them and thus to obtain the corresponding Green' s functions. A superfluid system is characterized by the fact that the wave function of the condensate ([7]), $\langle\Psi(t,\mathbf{r})\rangle = Re^{i\chi}$, is nonzero even for vanishingly small η, while the superfluid velocity is determined by the relation $\mathbf{v}_s = \frac{1}{m}(\nabla\chi - \mathbf{A})$. It can be shown ([2]) that the basic hydrodynamic quantities ρ_n, ρ_s, $\mathbf{v}_n$, $\mathbf{v}_s$, and θ can be expressed (not necessarily explicitly) through averages of certain combinations of second-quantized field operators in the Heisenberg representation. The equations of motion for these operators make it possible to obtain a system of hydrodynamic equations which, when linearized up to terms with "sources," coincides with the equations first obtained by I. M. Khalatnikov from phenomenological considerations.

considerations ([8]):

$$\frac{\partial\delta\rho}{\partial t} + \rho_s \operatorname{div}\mathbf{v}_s + \rho_n \operatorname{div}\mathbf{v}_n = i\sqrt{\rho_0}\,(\eta^* - \eta),$$

$$m\rho_s\frac{\partial v_s^\alpha}{\partial t} + m\rho_n\frac{\partial v_n^\alpha}{\partial t} = -\frac{\partial\delta P}{\partial r_\alpha} + \eta_2\sum_\beta\frac{\partial}{\partial r_\beta}\left(\frac{\partial v_n^\alpha}{\partial r_\beta} + \frac{\partial v_n^\beta}{\partial r_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\operatorname{div}\mathbf{v}_n\right) +$$

$$+\xi_1\frac{\partial}{\partial r_\alpha}\operatorname{div}\mathbf{v}_s + \xi_2\frac{\partial}{\partial r_\alpha}\operatorname{div}\mathbf{v}_n + b\,\frac{\partial\eta^*}{\partial r_\alpha} + b^*\,\frac{\partial\eta}{\partial r_\alpha} - \rho\left(\frac{\partial A^\alpha}{\partial t} + \frac{\partial U}{\partial r_\alpha}\right), \qquad (2)$$

$$m\frac{\partial v_s^\alpha}{\partial t} = -\frac{1}{\rho}\frac{\partial\delta\lambda}{\partial r_\alpha} + \xi_3\frac{\partial}{\partial r_\alpha}\operatorname{div}\mathbf{v}_s + \xi_4\frac{\partial}{\partial r_\alpha}\operatorname{div}\mathbf{v}_n + c\,\frac{\partial\eta^*}{\partial r_\alpha} + c^*\,\frac{\partial\eta}{\partial r_\alpha} - \left(\frac{\partial U}{\partial r_\alpha} + \frac{\partial A^\alpha}{\partial t}\right),$$

$$s\frac{\partial\delta\rho}{\partial t} + \rho\frac{\partial\delta s}{\partial t} + \rho s\operatorname{div}\mathbf{v}_n = \frac{\chi}{\theta}\Delta\delta\theta,$$

where

$$\delta P = \left(\frac{\partial P}{\partial\rho}\right)_\theta\delta\rho + \left(\frac{\partial P}{\partial\theta}\right)_\rho\delta\theta, \qquad \delta\lambda = -s\,\delta\theta + \frac{1}{\rho}\delta P, \qquad \rho = \rho_s + \rho_n,$$

ρ_0 is the condensate density in the state of static equilibrium, and the coefficients of thermal conductivity χ and viscosity η_2, ξ_i, b and c are constants, of which b and c are complex. To establish a certain relation between them it is convenient to consider the state of static equilibrium in the absence of external fields, but when η is a prescribed function of r:

$$U = 0, \qquad \mathbf{A} = 0, \qquad \eta = \eta(r), \qquad \theta = \text{const}, \qquad \mathbf{v}_n = 0, \qquad \mathbf{v}_s = \mathbf{v}_s(r).$$

The system (2) takes the form:

$$\rho_s \operatorname{div} \mathbf{v}_s = i\sqrt{\rho_0}(\eta^* - \eta),$$

$$\frac{\partial \delta P}{\partial r^*} = \xi_1 \frac{\partial}{\partial r_\alpha} \operatorname{div} \mathbf{v}_s + b\,\frac{\partial \eta^*}{\partial r_\alpha} + b^*\,\frac{\partial \eta}{\partial r_\alpha}, \tag{3}$$

$$\frac{\partial \delta P}{\partial r_\alpha} = \rho\xi_3 \frac{\partial}{\partial r_\alpha} \operatorname{div} \mathbf{v}_s + \rho c\,\frac{\partial \eta^*}{\partial r_\alpha} + \rho c^*\,\frac{\partial \eta}{\partial r_\alpha}.$$

Eliminating from (3) the terms with δP and $\mathbf{v}_s$, we obtain:

$$\left[b - \rho c - \frac{i\sqrt{\rho_0}}{\rho_s}(\rho\xi_3 - \xi_1)\right]\frac{\partial \eta^*}{\partial r_\alpha} + \left[b^* - \rho c^* + \frac{i\sqrt{\rho_0}}{\rho_s}(\rho\xi_3 - \xi_1)\right]\frac{\partial \eta}{\partial r_\alpha} = 0,$$

whence, by virtue of the arbitrariness of η, it follows that

$$b - \rho c = \frac{i\sqrt{\rho_0}}{\rho_s}(\rho\xi_3 - \xi_1). \tag{4}$$

We note that this relation between the coefficients differs from the relations in work ([3]), obtained by equating coefficients at identical terms in the last two equations (3)*.

To solve (2), we decompose the velocities into longitudinal and transverse components, pass to Fourier components, and, for convenience, denote the sought quantities $\delta\rho_k$, $\sum_\alpha k^\alpha v_s^\alpha(k)$, $\delta\theta_k$, $\sum_\alpha k^\alpha v_n^\alpha(k)$, respectively, by x_i $(i = 1, 2, 3, 4)$:

$$\omega x_1 - \rho_s x_2 - \rho_n x_3 = \sqrt{\rho_0}\,(\eta_k^* - \eta_{-k}),$$

$$k^2 \left(\frac{\partial P}{\partial \rho}\right)_\theta x_1 - (m\rho_s\omega + ik^2\xi_1)x_2 - (m\rho_n\omega + ik^2\gamma)x_3 + k^2\left(\frac{\partial P}{\partial \theta}\right)_\rho x_4 =$$

$$= k^2(b\eta^*_{-k} + b^*\eta_k - \rho U_k), \tag{5}$$

$$k^2\left(\frac{\partial P}{\partial \rho}\right)_\theta x_1-(m\rho\omega+ik^2\rho\xi_3)x_2-ik^2\rho\xi_4x_3+k^2\left[\left(\frac{\partial P}{\partial\theta}\right)_\rho-\rho s\right]x_4=$$

$$=k^2(\rho c\eta^*_{-k}+\rho c^*\eta_k-\rho U_k),$$

* This inaccuracy was independently corrected by Z. Galyasevich in ([10]).

$$\omega\left[\frac{s\theta}{\rho c_v}-\frac{\theta(\partial P/\partial\theta)_\rho}{\rho^2c_v}\right]x_1-\frac{s\theta}{c_v}x_3+\left(\omega+i\frac{k^2\chi}{\rho c_v}\right)x_4=0,$$

$$m\rho_n\omega v^\alpha_{n\tau}(k)=-ik^2\eta_2v^\alpha_{n\tau}(k)-\omega\rho_nA^\alpha_k,\qquad \gamma={}^4/{}_3\eta_2+\xi_2, \tag{5}$$

where $v^\alpha_{n\tau}(k)$ is the Fourier component of the transverse component of the normal velocity $\left(\sum_k k^\alpha v^\alpha_n(k)=0\right)$. From the last equation it follows that

$$v^\alpha_{n\tau}(k)=-\frac{\frac{1}{m}\omega}{\omega+ik^2\eta_2/m\rho_n}A^\alpha_k. \tag{6}$$

The solution of the remaining part of (5) can be represented in the form:

$$x_i=\Omega^{-1}\{\Phi_i(k,\omega)\eta_k-\varepsilon_i\Phi_i(k,-\omega)\eta^*_{-k}+\varphi_i(k,\omega)U_k\}^{1+\varepsilon_i}. \tag{7}$$

Here $\varepsilon_i=(-1)^i$; Ω is the determinant of system (5); Φ_i and φ_i are polynomials in ω and k^2 of degree not higher than the third, which it is not possible to write out here in full.

To find the Green' s functions one should use the theorem on the variation of the mean value (see, for example, (9)); here the role of the variation of the Hamiltonian is played by the terms with small external "sources" U, A, and η, which we choose in the form $g(t,r)=e^{-i\omega t+\varepsilon t+ikr}g_k+e^{i\omega t+\varepsilon t-ikr}g_{-k}$, where $\varepsilon\to 0+$, and g is any of the quantities U, A, η. In accordance with this theorem, for the Fourier images of the superfluid velocity and of the amplitude of the condensate wave function it is not difficult to obtain:

$$v^\alpha_{sl}(k)=\frac{\pi k^\alpha}{m\sqrt{\rho_0}}\left\{\langle\langle a_k-a^+_{-k};a^+_k\rangle\rangle^r_\omega\eta_k+\langle\langle a_k-a^+_{-k};a_{-k}\rangle\rangle^r_\omega\eta^*_{-k}\right.$$

$$\left.+V^{-1/2}\langle\langle a_k+a^+_{-k};\rho_{-k}\rangle\rangle^r_\omega U_k\right\},$$

$$R(k)=\pi\left\{\langle\langle a_k+a^+_{-k};a^+_k\rangle\rangle^r_\omega\eta_k+\langle\langle a_k+a^+_{-k};a_{-k}\rangle\rangle^r_\omega\eta^*_{-k}\right.$$

$$+V^{-1/2}\langle\langle a_k + a^+_{-k}; \rho_{-k}\rangle\rangle^r_\omega U_k\}\,, \qquad (8)$$

$$R(k) = \left(\frac{\partial R}{\partial \rho}\right)_\theta x_1 + \left(\frac{\partial R}{\partial \theta}\right)_\rho x_3,$$

where a^+_k, a_k are the particle creation and annihilation operators; ρ_k is the Fourier image of the density operator; V is the volume of the system, and the brackets $\langle\langle\cdots\rangle\rangle^r_\omega$ denote temporal Fourier images of retarded Green' s functions. Thus, the quantities x_i, on the one hand, are determined by the solution of the system of linearized hydrodynamic equations (7), and, on the other hand, relations of the type (8) can be written for them. Comparing these expressions makes it possible to obtain the Green' s functions in the so-called hydrodynamic approximation. For example, from (7) and (8) it follows that

$$\langle\langle a_k; a_{-k}\rangle\rangle^r_\omega = \frac{1}{\pi\Omega}\{F^*_\Phi(k,-\omega) - m\sqrt{\rho_0}\,\Phi^*_2(k,-\omega)\},$$

$$\langle\langle a_k; a^+_k\rangle\rangle^r_\omega = \frac{1}{\pi\Omega}\{F_\Phi(k,\omega) + m\sqrt{\rho_0}\,\Phi_2(k,\omega)\}, \qquad (9)$$

$$\langle\langle a_k; \rho_{-k}\rangle\rangle^r_\omega = \frac{V^{-1/2}}{\pi\Omega}\{F_\varphi(k,\omega) + m\sqrt{\rho_0}\,\varphi_2(k,\omega)\},$$

where

$$F_f(k,\omega) = (\partial R/\partial\rho)_\theta f_1(k,\omega) + (\partial R/\partial\theta)_\rho f_3(k,\omega); \qquad f_i = \Phi_i, \varphi_i.$$

Let us also calculate the tangential component of the tensor Green' s function current-current $\langle\langle j^\alpha_\tau(k); j^\beta_\tau(-k)\rangle\rangle^r_\omega$. In accordance with the definition of the current in the presence of a vector field,

$$m\rho_n v^\alpha_{n\tau} = \delta\langle j^\alpha_\tau(k)\rangle - \rho_n A^\alpha_k,$$

and it remains only to find the expression for the variation $\delta\langle j^\alpha_\tau(k)\rangle$ due to the adiabatically switched-on perturbation

$$\delta H = -\frac{1}{m}\int \mathbf{j}_\tau(t,r)\mathbf{A}(t,r)\,dr.$$

Using the theorem on the variation of the mean and relation (6), we obtain

$$\langle\langle j_\tau^\alpha(k); j_\tau^\beta(-k)\rangle\rangle_\omega^r = -\frac{V}{2\pi}\frac{ik^2\eta_2}{\omega + ik^2\eta_2/m\rho_n}\left(\delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right). \qquad (10)$$

In an analogous way one can find the Green functions $\langle\langle\rho_k;\rho_{-k}\rangle\rangle_\omega^r$, $\langle\langle j_l^\alpha(k);\rho_{-k}\rangle\rangle_\omega^r$, $\langle\langle j_l^\alpha(k); j_l^\beta(-k)\rangle\rangle_\omega^r$. It should be noted that the expressions found for the Green functions are asymptotic and are valid only for $k \ll 1/L$, $\omega \ll 1/T$ (L, T are the mean free path and the relaxation time), since the hydrodynamic equations themselves are valid only for sufficiently "slow" changes of the hydrodynamic quantities.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.