THE DISTANCE BETWEEN "NEIGHBORING" PRIME IDEALS

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Abstract

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THE DISTANCE BETWEEN "NEIGHBORING" PRIME IDEALS

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Analyzing the classical proof ([1]) of the theorem on the difference between "neighboring" prime numbers, it is easy to see that it is based on the knowledge:

a) of the absence of zeros of $\zeta(\sigma+it)$ in the region $\sigma \geqslant 1 - A/\ln^a(|t|+2)$ $(a<1)$,

b) of an estimate of $N(\sigma, T)$—the number of zeros $\rho = \beta + i\gamma$ of the function $\zeta(\sigma+it)$ in the region $\beta \leqslant \sigma;\ 0 \leqslant \gamma \leqslant T$.

With the aid of I. M. Vinogradov' s method for estimating trigonometric sums ([2]), we prove here Theorems 1 and 2.

Theorem 1. *The Dedekind zeta-function $\zeta_K(\sigma+it)$ of a field of algebraic numbers K of degree n has no zeros in the region*

$$\sigma \geqslant 1 - A_1/\ln^{2/3}(|t|+2),$$

where $A_1 > 0$ depends only on the field K (cf. ([3,4])).

Theorem 2. $\zeta_K(1/2+it) \ll |t|^{n/4 - c/n^2 \ln n}$; *$c$ is an absolute constant.*

From Theorem 1, by means of a simple generalization of Hoheisel' s method ([1]), we obtain the following theorem.

Theorem 3. *Let, as usual, $\pi(x)$ denote the number of prime ideals of the field K with norm not exceeding x. Suppose further that*

$$N_K(\sigma, T) \ll T^{b(1-\sigma)} \ln^{c_1} T.$$

Then, for $\theta > 1 - 1/b$,

$$\pi(x + x^\theta) - \pi(x) \sim x^\theta / \ln x.$$

Corollary. *Since from the estimate*

$$\zeta_K(1/2+it) \ll |t|^{c_0} \ln^{c_2} |t|$$

it follows that

$$N_K(\sigma, T) \ll T^{2(1+2c_0)(1-\sigma)} \ln^{c_3} T, \tag{5}$$

Theorems 2 and 3 give

$$\theta > \frac{1+4c_0}{2+4c_0} = 1 - \frac{1}{n+2-4c/n^2 \ln n}.$$

We outline the proof of Theorems 1 and 2. In the paper ([4]) it is shown that the question of shifting the zeros of $\zeta_K(\sigma+it)$ reduces to estimating the sum

$$S = \sum_{\substack{a<a_i<a' \\ (a_1,\dots,a_n)\in K_1^X \setminus K_0^X}} e^{2\pi i F(a_1,\dots,a_n)}. \tag{1}$$

Here

$$F(a_1,\dots,a_n) = -\frac{t}{2\pi} \ln \prod_{j=1}^{n} (a_1\alpha_1^{(j)} + \dots + a_n\alpha_n^{(j)});$$

a_i are rational integers; $\alpha_1,\dots,\alpha_n$ is a basis of some integral ideal of the field K; $\alpha_i^{(j)}$ are the conjugates of α_i; K_1^X (respectively K_0^X) is the set of tuples $(a_1,\dots,a_n)$ such that $|a_r| \leqslant 2X$ (respectively X) $(r=1,\dots,n)$, and the image in R^n of the number of the field K

$$a = a_1\alpha_1 + \dots + a_n\alpha_n$$

belongs to the fundamental region of the field (for more detail see ([4])).

Lemma 1. *For any integer $m \geqslant 1$ and any tuple*

$$(a_1,\dots,a_n) \in K_1^X \setminus K_0^X$$

$$|\partial^m F(a_1,\dots,a_n)/\partial a_i^m| \leqslant c_4^m (m-1)!\, tX^{-m}. \tag{2}$$

For any fixed a_r $(r \neq i)$ satisfying $(a_1,\dots,a_n) \in K_1^X \setminus K_0^X$ and any integer $m_1 \geq 1$, the interval of variation of a_i can be divided into $\leq c_5^{m_1}$ intervals, for each of which there exists an integer m $(m_1 \leq m \leq m_1+n)$ such that the inequality

$$|\partial^m F(a_1,\dots,a_n)/\partial a_i^m| \geq c_6^m (m-1)! tX^{-m} \tag{3}$$

holds for all points of this interval.

Proof. It is easily computed that

$$\frac{\partial^{m_1} F(a_1,\dots,a_n)}{\partial a_i^{m_1}} = (-1)^{m_1}(m_1-1)!t \sum_{j=1}^{n} \left[\frac{\alpha_i^{(j)}}{a_1\alpha_1^{(j)} + \dots + a_n\alpha_n^{(j)}}\right]^{m_1} =$$

$$= \frac{(-1)^{m_1}(m_1-1)!t}{\left|a_1\alpha_1^{(d)}+\dots+a_n\alpha_n^{(d)}\right|^{m_1}}\sum_{j=1}^{n}\left[\frac{\alpha_i^{(j)}}{\left|\alpha_i^{(d)}\right|}\frac{\left|a_1\alpha_1^{(d)}+\dots+a_n\alpha_n^{(d)}\right|}{a_1\alpha_1^{(j)}+\dots+a_n\alpha_n^{(j)}}\right]^{m_1},$$

where

$$\left|\frac{\alpha_i^{(d)}}{a_1\alpha_1^{(d)}+\dots+a_n\alpha_n^{(d)}}\right| = \max_{1\le l\le n}\left|\frac{\alpha_i^{(l)}}{a_1\alpha_1^{(l)}+\dots+a_n\alpha_n^{(l)}}\right|.$$

With the help of Turán's theorem 2 ([6]) and lemma 1 ([4]), we obtain the existence of m ($m_1 \le m \le m_1+n$) such that

$$\left|\frac{\partial^m F(a_1,\dots,a_n)}{\partial a_i^m}\right| \ge \frac{c_7^m}{m^n}(m-1)!tX^{-m}$$

for any fixed set $(a_1,\dots,a_n) \in K_1^X \setminus K_0^X$. Hence the assertions of the lemma follow immediately ([4]).

Basic lemma. Let $m = [\ln t/\ln X]+1$ and $e^{\ln^{2/3} t} < X < Bt^{(n+1)/n}$. Then

$$\left|\sum_{\substack{a\le a_i\le a' \\ (a_1,\dots,a_n)\in K_1^X\setminus K_0^X}} e^{2\pi i F(a_1,\dots,a_n)}\right| < CX^{1-\gamma/m^2},$$

where C and γ depend only on the field K.

The proof almost literally repeats the arguments of I. M. Vinogradov ([2]). We note the main specific points.

For $m > n+4$ put $Y = [X^{1/3}]$; x and y run through the values $1,2,\dots,Y$; $m_0 = 3m$; $r = 2b$; $b = lm_0 + [m_0(m_0+1)/2+1]$.

Replacing a_i in (1) by $a_i + xy$, we obtain, by virtue of inequality (2),

$$|S| \le \frac{1}{Y^2}\sum_{\substack{a\le a_i\le a' \\ (a_1,\dots,a_n)\in K_1^X\setminus K_0^X}} |S_{a_i}| + 2X^{2/3},$$

where

$$S_{a_i} = \sum_x\sum_y e^{2\pi i(A_1xy+\dots+A_{m_0}x^{m_0}y^{m_0})},$$

$$A_s = \frac{1}{2\pi s!}\frac{\partial^{sF}(a_1, \dots, a_n)}{\partial a_i^s}.$$

As in ([2]), the number ν of points $(\{A_1\eta_1\}, \dots, \{A_{m_0}\eta_{m_0}\})$ falling into the given small domain of theorem 5 ([2]) is estimated by the quantity

$$(2bc_6^m)^{m_0} X^{m_0(m_0+1)/2} \prod_{[r/2m]+1\le m_r\le 3m-3} X^{m_r-3m+3} \prod_{m\le m_r\le [r/2m]-2} X^{3m-2-3m_r} <$$

$$< (2bc_6^m)^{m_0} X^{m_0(m_0+1)/2}(1-\delta),$$

where m_r are those values of m for which inequality (3) holds; $\delta > 0$ depends only on the field.

Choosing k and l sufficiently large in Theorems 5 and 1 [2], we obtain

$$|S| < CX^{1-\gamma/m^2}.$$

For $m < n + 4$, the estimate for $|S|$ given in [4] is sufficient. Theorem 1 is now obtained in the usual way [1,4].

With the aid of an estimate for the sum (1) in the interval $c_7 t^{1/3} < X < c_8 t^{1/2}$, easily obtained by the method of I. M. Vinogradov [7,8], and of the "approximate functional equation" for $\zeta_K(\sigma + it)$ [9], Theorem 2 is proved.

Although in special cases (for example, a purely real field K) the methods of H. Weyl or van der Corput are easily applied to estimating $\zeta_K(1/2 + it)$ (which gives, in Theorems 2 and 3, $c = 1/12$ and $\delta > (3n + 2)/(3x + 5)$), it is not clear how these methods can be used in the case of an arbitrary field.

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Note: Figure translations are in progress. See original paper for figures.