A HARNACK INEQUALITY FOR GENERALIZED SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS OF SECOND ORDER

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Abstract

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A HARNACK INEQUALITY FOR GENERALIZED SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS OF SECOND ORDER

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In the works (^{1},^{2}) J. Moser proposed a method for obtaining the Harnack inequality for generalized solutions (g.s.) from the classes $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{W}{}_2^1(Q)$ of the equations $(\mathcal{A}_{ij}u_{x_i})_{x_j} = 0$ and $u_t = (\mathcal{A}_{ij}u_{x_i})_{x_j}$. In (^{3}) J. Serrin extended the result of (^{1}) to g.s. of a certain class of quasilinear elliptic equations. Here we extend the Harnack inequality to g.s. of a certain class of quasilinear parabolic equations of second order (see (^{8})), and in the proof the approaches used in ($^{1-3}$)* are developed.

Let Ω be some bounded n-dimensional domain in E_n, Q the cylinder $\Omega \times [t_1, t_2]$. By $L_{p,p_1}(Q)$ $(p, p_1 \geq 1)$ we denote the space of all measurable functions in Q for which the norm is finite

$$\|u\|_{p,p_1,Q} = \left(\int_{t_1}^{t_2} \|u\|_{p,\Omega}^{p_1} \, dt \right)^{1/p_1} = \left(\int_{t_1}^{t_2} \left(\int_{\Omega} |u|^p \, dx \right)^{p_1/p} dt \right)^{1/p_1}. \quad (1)$$

For $p_1 = \infty$,

$$\|u\|_{p,\infty,Q} = \operatorname{vrai\,max}_{t \in [t_1,t_2]} \|u\|_{p,\Omega}.$$

We introduce also the following notation. $\overset{\circ}{W}{}_2^{1,0}(Q)$ is the closure of the set of all smooth and finite in Q functions in the norm $(\|u\|_{2,Q}^2 + \|u_x\|_{2,Q}^2)^{1/2}$; $\overset{\circ}{W}{}_2^{1,1}(Q)$ is the subset of all functions from $\overset{\circ}{W}{}_2^{1,0}(Q)$ which vanish in some neighborhood of the lateral surface and the lower base of the cylinder Q and have the generalized derivative $u_t \in L_2(Q)$; $U_2^{1,0}(Q)$ is the space of all functions $u(x,t)$ belonging to $L_2(Q)$ and having generalized derivatives u_{x_i} from $L_2(Q)$ and a finite $\operatorname{vrai\,max}_{t \in [t_1,t_2]} \|u\|_{2,\Omega}$, with norm

$$\langle\!\langle u \rangle\!\rangle_Q = \left(\operatorname{vrai}\max_{t\in[t_1,t_2]} \|u\|^2_{2,\Omega} + \|u_x\|^2_{2,Q}\right)^{1/2}, \tag{2}$$

where

$$u_x = (u_{x_1}, \dots, u_{x_n}), \qquad \|u_x\|_{2,Q} = \left\| \left(\sum_{i=1}^{n} u^2_{x_i} \right)^{1/2} \right\|_{2,Q}.$$

Denote by $K_\rho(x_0)$ the n-dimensional ball of radius ρ ($\rho > 0$) with center at the point x_0, and by $Q_\rho(x_0, t_0)$ the cylinder $K_\rho(x_0) \times [t_0 - \rho^2, t_0]$.

Lemma 1. Let $u(x,t) \in U_2^{1,0}(Q_\rho)$, and let the numbers l, l_1 $(l, l_1 \geq 1)$ satisfy the conditions

$$1/l + 2/nl_1 \geq 1/2,$$

$$l \geq 2;\ l_1 \geq 2 \text{ for } n \geq 3; \quad 2 \leq l < \infty,\ l_1 > 2 \text{ for } n = 2; \quad l \geq 2,\ l_1 \geq 4 \quad \text{for } n = 1. \tag{3}$$

Then the function $u(x,t)$ belongs to the space $L_{l,l_1}(Q_\rho)$ and satisfies the inequality

$$\|u\|_{l,l_1,Q_\rho} \leq a_0 \left(\operatorname{vrai}\max_{t\in[t_1,t_2]} \|u\|_{2,K_\rho}\right)^{1-2/l_1} \left(\|u\|_{2,Q_\rho} + \|u_x\|_{2,Q_\rho}\right)^{\varkappa} \left(\|u\|_{2,Q_\rho}\right)^{2/l-\varkappa}, \tag{4}$$

where $\varkappa = n/2 - n/l$ (obviously, $\varkappa \in [0, 2/l_1]$) and a_0 depends only on n for $n \neq 2$ and on l for $n = 2$.

In the case $1/l + 2/nl_1 > 1/2$, inequality (4) implies the inequality

$$\|u\|_{l,l_1,Q_\rho} \leq \varepsilon \langle u \rangle_{Q_\rho} + a_0' \, \varepsilon^{-\frac{1}{\varkappa - 2/l_1} \frac{1+\varkappa-2/l_1}{\varkappa-2/l_1}} \|u\|_{2,Q_\rho}, \tag{5}$$

where ε is an arbitrary positive number.

Consider the equation

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* A certain generalization of the result of Moser' s work (2) was obtained in (4), where it was shown that the Harnack inequality will also hold for a somewhat broader class of g.s. of the equation $u_t = (\mathcal{A}_{ij} u_{x_i})_{x_j}$ than $\overset{\circ}{W}{}^1_2(Q)$.

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\mathcal{L}_i(x,t,u,u_x) + \mathcal{L}_0(x,t,u,u_x) = 0\ *, \tag{6}$$

where $\mathcal{L}_i(x,t,u,p)$ and $\mathcal{L}_0(x,t,u,p)$ $(p = (p_1,\dots,p_n))$ are given functions of the variables x,t,u,p, defined in the domain

$$\mathcal{R}:\ x \in \Omega,\ t \in [0,T],\ -\infty < u < \infty,\ -\infty < p_k < \infty,\ k = 1,\dots,n,$$

measurable in x,t in the cylinder $Q_T = \Omega \times [0,T]$ for fixed u and p, and continuous in the variables u and p for fixed x and t. Suppose that for all $(x,t,u,p) \in \mathcal{R}$ the inequalities

$$\left[\sum_{i=1}^{n} \mathcal{L}_i^2(x,t,u,p)\right]^{1/2} \le \mu|p| + \mathcal{A}|u| + \mathcal{F},$$

$$|\mathcal{L}_0(x,t,u,p)| \le \mathcal{B}|p| + \mathcal{C}|u| + \mathcal{G}, \tag{7}$$

$$\mathcal{L}_i(x,t,u,p)p_i \ge \nu|p|^2 - \mathcal{D}|u|^2 - \mathcal{H},$$

hold, where ν and μ are positive constants; $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{F},\mathcal{G},\mathcal{H}$ are nonnegative functions of the variables x,t, belonging to the spaces

$$\mathcal{A},\mathcal{B},\mathcal{F} \in L_{q,\infty}(Q_T), \qquad \mathcal{C},\mathcal{D},\mathcal{G},\mathcal{H} \in L_{q/2,\infty}(Q_T), \qquad q > n \ge 2. \tag{8}$$

A generalized solution from the class $\mathring{W}_2^{1,0}(Q_T)$ of equation (6), whose coefficients satisfy conditions (7), (8), is a function

$$u(x,t) \in U_2^{1,0}(Q_T),$$

satisfying, for almost all $t_1, t_2 \in [0,T]$ and all

$$\Phi \in \mathring{W}_2^{1,1}(Q), \qquad Q = \Omega \times [t_1,t_2],$$

the integral identity

$$\int_\Omega u\Phi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left[-u\Phi_t + \mathcal{L}_i(x,t,u,u_x)\Phi_{x_i} + \mathcal{L}_0(x,t,u,u_x)\Phi\right]\,dx\,dt = 0. \tag{9}$$

Denote by Γ_T the set of all points (x,t) lying on the lateral surface or on the lower base of the cylinder Q_T, and let

$$\widehat{Q}_{2r} = \widehat{Q}_{2r}(x_0,t_0) = Q_{2r}(x_0,t_0+2r^2) = K_{2r}(x_0) \times [t_0-2r^2, t_0+2r^2].$$

With the aid of the main lemma of [2] and Lemma 1 one proves

Theorem 1. Let $u(x,t)$ be a generalized solution from the class $U_2^{1,0}(Q_T)$ of equation (6), whose coefficients satisfy conditions (7), (8). Suppose that this generalized solution is nonnegative in the cylinder $\widehat{Q}_{2r}(x_0,t_0)$, contained in Q_T and at a positive distance from Γ_T. Then the function $u(x,t)$ satisfies the inequality (Harnack)

$$\operatorname{vrai\,max}_{Q^-_{r/4}} u \le \mathrm{const}\left(\operatorname{vrai\,min}_{Q^+_{r/4}} u + k_r\right), \tag{10}$$

where

$$Q^-_{r/4} = Q_{r/4}(x_0, t_0 - \tfrac{7}{4}r^2), \qquad Q^+_{r/4} = Q_{r/4}(x_0, t_0+2r^2),$$

$$k_r = r^\delta \|\mathcal{F}\|_{q,\infty,\widehat{Q}_{2r}} + r^{2\delta}\|\mathcal{G}\|_{q/2,\infty,\widehat{Q}_{2r}} + r^\delta\left(\|\mathcal{H}\|_{q/2,\infty,\widehat{Q}_{2r}}\right)^{1/2}, \qquad \delta = 1-\frac{n}{q}.$$

The constant in (10) depends only on

$$n, \nu, \mu, q, \ r^\delta\|\mathcal{A},\mathcal{B}\|_{q,\infty,\widehat{Q}_{2r}}$$

and

$$r^{2\delta}\|\mathcal{C},\mathcal{D}\|_{q/2,\infty,\widehat{Q}_{2r}}.$$

As a consequence of Harnack' s inequality one easily obtains an estimate of the Hölder constant of an arbitrary generalized solution from the class $\mathcal{V}_2^{1,0}(Q_T)$ of equation (6), considered under conditions (7), (8).

In the case

$$\mathcal{L}_i(x,t,u,p) = \sum_{j=1}^n L_{ij}p_j + A_i u + F_i, \qquad \mathcal{L}_0(x,t,u,p) = \sum_{j=1}^n B_{jp} j +$$

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* By $\dfrac{\partial}{\partial x_i}\mathcal{L}_i(x,t,u,u_x)$ is meant the total derivative of the function $\mathcal{L}_i(x,t,u(x,t),u_x(x,t))$ with respect to the variable x_i.

$$+Cu+G$$

the estimate of the Hölder constant was first obtained by different methods by O. A. Ladyzhenskaya and N. N. Ural’ tseva in $({}^6)$, and (for $A_i = B_i = F_i = C = G = 0$) by J. Nash in $({}^5)$.

Theorem 2. Let $u(x,t)$ be an arbitrary generalized solution from the class $\mathcal{V}_2^{1,0}(Q_T)$ of equation (6), satisfying conditions (7), (8), and let $Q' = \Omega' \times [\delta', T]$, $Q'' = \Omega'' \times [\delta'', T]$, $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$, $\delta' > \delta'' > 0$. Then the function $u(x,t)$ has a finite $\operatorname{vrai\,max}_{Q'} |u|$ and, for almost all (x,t), $(\bar{x},\bar{t}) \in Q'$, satisfies the inequality

$$|u(x,t) - u(\bar{x},\bar{t})| \leqslant \text{const}\left(\operatorname{vrai\,max}_{Q'} |u| + K\right)\left(|x-\bar{x}|^2 + |t-\bar{t}|\right)^{\gamma}, \tag{11}$$

where $\gamma \in (0,1]$,

$$K = \|\mathcal{F}\|_{q,\infty,Q''} + \|\mathcal{G}\|_{q/2,\infty,Q''} + \left(\|\mathcal{H}\|_{q/2,\infty,Q''}\right)^{1/2},$$

and the constant in (11) depends only on n, ν, μ, q, the norms $\|\mathcal{A},\mathcal{B}\|_{q,\infty,Q''}$, $\|\mathcal{C},\mathcal{D}\|_{q/2,\infty,Q''}$, on the geometry of Q', and on the distance from Q' to the lateral surface and the lower base of the cylinder Q''.

The functions $\mathcal{A}, \mathcal{B}, \dots, \mathcal{H}$ may be regarded as elements of spaces $L_{p,p_1}(Q_T)$ of general form ($p, p_1 \geqslant 1$). In this case, for generalized solutions of the quasilinear equation (6) satisfying conditions (7), theorems (see $({}^8)$) analogous to Theorems 1 and 3-7 of paper $({}^7a)$, concerning linear parabolic equations, are proved.

Remark added in proof.

1. The results of Theorems 1 and 2 remain valid if the functions $\mathcal{A}, \mathcal{B}, \dots, \mathcal{H}$ are characterized in terms of belonging to spaces $L_{p,p_1}(Q_T)$ of general form, i.e., if instead of (8) one imposes the condition

$$\mathcal{A}, \mathcal{B}, \mathcal{F} \in L_{2p,2p_1}(Q_T), \qquad \mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H} \in L_{p,p_1}(Q_T)$$
$$n/p + 2/p_1 < 2, \qquad p \geqslant 1, \quad p_1 \geqslant 1, \quad n \geqslant 1. \tag{8'}$$

2. In the case of bounded generalized solutions the results of these theorems remain valid if, instead of (7), one assumes that

$$\left[\sum_{i=1}^{n} \mathcal{L}_i^2\right]^{1/2} \leqslant \mu|p| + \mathcal{F}, \qquad |\mathcal{L}_0| \leqslant \mu_1|p|^2 + \mathcal{G}, \qquad \mathcal{L}_i p_i \geqslant \nu|p|^2 - \mathcal{H}, \tag{7'}$$

where $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ satisfy (8’).

3. During the International Congress of Mathematicians in Moscow in 1966 it became known to us that, recently, a result analogous to Theorem 1 in the case of conditions (7), (8^1) had been obtained by D. G. Aronson and J. Serrin (an announcement of their result appears in *Notices of the*

Am. Math. Soc., 13 April 1966, p. 381), and, in the case of bounded generalized solutions under conditions (7^1), $(8')$, also by N. S. Trudinger. In addition, for linear equations Harnack' s inequality has been extended by L. P. Kuptsov to a certain class of non-hyperbolic equations. These results have not yet been published.

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Note: Figure translations are in progress. See original paper for figures.

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