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ON SOME TWO-DIMENSIONAL SELF-SIMILAR FLOWS OF A POLYTROPIC GAS WITH VARIABLE ENTROPY

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Let us consider plane nonisentropic flows of a polytropic gas with the equation of state $p = a^2(s)\rho^\gamma$ (p is pressure, s is entropy, ρ is density, γ is the adiabatic exponent), depending on the self-similar variables $\xi_i = x_i/t$ and determined by the system of equations:

$$(l_1 + b\rho_1 + cs_1)\frac{\partial u_1}{\partial \xi_1} + l_2\frac{\partial u_1}{\partial \xi_2} + (b\rho_2 + cs_2)\frac{\partial u_2}{\partial \xi_1} = 0, \tag{1}$$

$$(b\rho_1 + cs_1)\frac{\partial u_1}{\partial \xi_2} + l_2\frac{\partial u_2}{\partial \xi_1} + (l_2 + b\rho_2 + cs_2)\frac{\partial u_2}{\partial \xi_2} = 0,$$

$$(l_1\rho_1 + \rho)\frac{\partial u_1}{\partial \xi_1} + l_2\rho_1\frac{\partial u_1}{\partial \xi_2} + l_1\rho_2\frac{\partial u_2}{\partial \xi_1} + (l_2\rho_2 + \rho)\frac{\partial u_2}{\partial \xi_2} = 0; \tag{2}$$

$$l_1 s_1\frac{\partial u_1}{\partial \xi_2} + l_2 s_1\frac{\partial u_1}{\partial \xi_2} + l_1 s_2\frac{\partial u_2}{\partial \xi_1} + l_2 s_2\frac{\partial u_2}{\partial \xi_2} = 0;$$

here $l_i = u_i - \xi_i,\ \rho_i = \partial\rho/\partial u_i,\ s_i = \partial s/\partial u_i\ (i = 1, 2)$, u_i are the components of the velocity vector $\mathbf{u}$, x_i are Cartesian coordinates, t is time,

$$b = \gamma a^2(s)\rho^{\gamma-2}, \qquad c = 2a(s)a'(s)\rho^{\gamma-1}. \tag{4}$$

It is also assumed that u_1 and u_2 are functionally independent. Then equations (1) correspond to the equations of motion, (2) to the equation of continuity, and (3) to the adiabaticity condition.

Within the framework of this class of flows one can solve a number of important gas-dynamic problems; in particular, problems of the diffraction of plane shock waves by a convex corner and problems of the irregular reflection of plane shock waves from oblique walls. In the present note a closed system of equations is derived for the functions $\rho(u_1, u_2)$, $s(u_1, u_2)$ in the hodograph plane. These equations can be used to carry out various kinds of linearization and to construct approximate theories. A class of exact solutions is obtained in the presence of shock waves. The system of equations for $\rho(u_1, u_2)$ and $s(u_1, u_2)$ is also used to derive an approximate system of "short-wave" equations (see (^{1})), valid in a narrow zone behind a curvilinear shock wave, in which the gradients $|\mathbf{u}|$ and p are large.

1. The principal possibility of obtaining a system of equations for ρ and s in the hodograph plane is clear. Indeed, performing the hodograph transformation for the system of equations (1)–(3) and calculating the determinant of the matrix of the resulting system, homogeneous with respect to $\partial\xi_i/\partial u_k$, we shall have a relation of the form $F(\xi_1, \xi_2, u_1, u_2, \rho, s, \rho_i, s_i) = 0$. Replacing one of the equations of the system (1)–(3) by this condition and finding from it the derivatives $\partial\xi_1/\partial u_k$, we arrive at a system of 3 equations of the form

$$a_{11}^{(j)} \frac{\partial \xi_2}{\partial u_1} + a_{(12)}^{(j)} \frac{\partial \xi_2}{\partial u_2} + a_{(13)}^{(j)} = 0, \qquad j = 1, 2, 3, \tag{1.1}$$

where $a_{ik}^{(j)}$ are functions of ξ_2, u_1, u_2 (the $a_{ik}^{(j)}$ contain second derivatives of ρ and s).

Equating to zero the third-order determinant for (1.1), we find ξ_2 as a function of u_1, u_2, and, finally, substituting the derivatives $d\xi_2/du_k$ from the relation obtained into any two equations of the system (1.1), we arrive at a system of 2 partial differential equations of third order for ρ and s.

However, a direct implementation of this algorithm is quite difficult. The equations for ρ and s can be obtained only after establishing several auxiliary algebraic relations between the sought quantities, the principal one of which has the form

$$l_2 s_1 - l_1 s_2 + b(s_1 \rho_2 - s_2 \rho_1) = 0. \tag{1.2}$$

Finally, the system of equations for ρ and s can be written in the form

$$Q[\rho(1 + R^2) - b(\rho_1 - \rho_2 R)^2](1 - P_2) +$$

$$S[\rho(RP + T - (b\rho_2 + cs_2)R) + P(b\rho_2 + cs_2)(\rho_1 - \rho_2 R)] = 0, \tag{1.3}$$

$$Q[\rho(1 + R^2) - b(\rho_1 - \rho_2 R^2)]P_1 + S[\rho(P + (b\rho_1 + cs_1)R) -$$

$$-P(b\rho_1 + cs_1)(\rho_1 - \rho_2 R))] = 0,$$

where $R = s_1/s_2,\ R_i = \partial R/\partial u_i,\ P_i = \partial P/\partial u_i,\ T_i = \partial T/\partial u_i,\ Q = R_1(b\rho_2 +$

$$+cs_2) - R_2(b\rho_1 + cs_1), \quad P = \frac{1}{Q}[(b\rho_1 + cs_1)(R + T_2) + (b\rho_2 + cs_2)(1-$$

$$-T_1)], \quad T = b(\rho_2 R - \rho_1), \quad S = RR_1 + R_2 + T_2 R_1 - T_1 R_2.$$

The flow in the plane of the self-similar variables ξ_1, ξ_2 is determined by the relations

$$\xi_1 = u_1 - RP - T, \qquad \xi_2 = u_2 - P. \tag{1.4}$$

Any three-times continuously differentiable solution of the system (1.3) satisfying, in some domain, the conditions

$$Q = 0, \quad T \neq 0, \quad (1 - P_2)(1 - R_1 P - T_1) - P_1(R + R_2 P + T_2) \neq 0, \tag{1.5}$$

satisfies the system of equations (1)–(3) and, consequently, gives a solution of the equations of gas dynamics for the case under consideration; here u_1 and u_2, as functions of ξ_1, ξ_2, are found from (1.4).

2. We shall seek particular solutions of the system (1.3) in the form

$$\rho = u_1^{\alpha} u_2^{\lambda} f_\rho(u_1 u_2^{\chi}), \qquad a^2(s) = u_1^{\beta} u_2^{\mu} f_s(u_1 u_2^{\chi}). \tag{2.1}$$

An analysis of the dimensions of the terms in the system (1.3) shows that the system (1.3) has nontrivial solutions of the form (2.1) only under the conditions

$$\lambda = \mu = 0, \qquad \chi = -1, \qquad (\gamma - 1)\alpha + \beta - 2 = 0; \tag{2.2}$$

for f_ρ and f_s one obtains a system of two ordinary equations with independent variable u_1/u_2, while the equations (1.4) will have the form $\xi_i = u_{iF}i(u_1/u_2)$, where F_1 and F_2 are expressed in terms of the functions f_ρ and f_s. Thus, returning to the variables ξ_i and setting

$$u_1 = \xi_1 f_1(\xi), \qquad u_2 = \xi_1 f_2(\xi), \qquad \rho = \xi_1^{\alpha} f_3(\xi),$$

$$a^2(s) = \xi_1^{\beta} f_4(\xi), \qquad \xi = \xi_1/\xi_2, \tag{2.3}$$

for the functions f_1 we shall have the system of ordinary equations

$$\xi(f_1 - f_2\xi)f_1' + \gamma\xi f_3^{\gamma-2} f_4 f_3' + \xi f_3^{\gamma-1} f_4' -$$

$$-f_1' + f_1^2 + (\gamma\alpha + \beta) f_3^{\gamma-1} f_4 = 0,$$

$$\xi(f_1 - f_2\xi)f_2' - \gamma\xi^2 f_3^{\gamma-2} f_4 f_3' - \xi^2 f_3^{\gamma-1} f_4' - f_2 + f_1 f_2 = 0, \tag{2.4}$$

$$\xi f_3 f_1' - \xi^2 f_3 f_2' + \xi(f_1 - f_2\xi) f_3' - \alpha f_3 + (\alpha+1) f_1 f_3 = 0,$$

$$\xi(f_1 - f_2\xi) f_4' - \beta f_4 + \beta f_1 f_4 = 0.$$

Here the prime denotes differentiation with respect to ξ.

The class of solutions (2.3) permits the construction of flows with discontinuities of the oblique-shock type, specified by the equation

$$q_1\xi_1 + q_2\xi_2 + q_3 = 0, \qquad q_i = \text{const}, \tag{2.5}$$

when the flow in front of and behind the shock front belongs to the type under consideration. In this case all the Hugoniot conditions along (2.4) are satisfied when (2.2) is fulfilled exactly.

Flows of the form (2.3) have stationary streamlines, and therefore the solutions of system (2.4) can be used to construct flows in plane curvilinear channels of special form with a bend in the walls, when a shock wave appears at the bend.

In the case $\alpha = -2$, $\beta = 2\gamma$, the pressure p does not depend on time, and to a flow of the form (2.3) one may apply Smith' s theorem (^{2}), by means of which one obtains a class of solutions with one arbitrary function of one argument of the form

$$u_1 = \frac{mx_1}{mt + h(m)} f_1\left(\frac{x_1}{x_2}\right), \qquad u_2 = \frac{mx_1}{mt + h(m)} f_2\left(\frac{x_1}{x_2}\right),$$

$$\rho = \frac{[mt + h(m)]^2}{m^2 x_1^2} f_3\left(\frac{x_1}{x_2}\right), \qquad s = \frac{x_1^{2\gamma}}{m^2[mt + h(m)]^{2\gamma}} f_4\left(\frac{x_1}{x_2}\right), \tag{2.6}$$

$$m = x_2 \exp\left(\int \frac{d\xi}{\xi + \varphi(\xi)}\right), \qquad \varphi(\xi) = \frac{f_1(\xi)}{f_2(\xi)},$$

where $h(m)$ is an arbitrary function.

3. Let us consider a shock wave whose motion is determined by the equation $\Phi(\xi_1, \xi_2) = 0$, propagating through a stationary gas of constant density. Let the flow behind the wave depend on ξ_1, ξ_2, and suppose that in the ξ_1, ξ_2 plane, beyond the curve $\Phi(\xi_1, \xi_2) = 0$, there is a narrow region in which the gradients of the velocity modulus and pressure are large—the so-called region of "short" waves. For weak shock waves, under the assumption that the flow is isentropic, the equations of "short" waves were derived in (^{1}). We shall assume that in the mentioned zone behind the shock wave the relations

$$(\mathbf{u} \cdot \vec{\tau}) = \varepsilon(\mathbf{u} \cdot \mathbf{n}), \tag{3.1}$$

$$\partial|\mathbf{u}|/\partial\tau = \lambda\, \partial|\mathbf{u}|/\partial n, \tag{3.2}$$

are satisfied, where $\mathbf{n}$ is the unit normal vector to the shock wave, $\vec{\tau}$ is the unit tangent vector, and $|\varepsilon| \ll 1$, $|\lambda| \ll 1$. Note that (3.2) entails the condition $\partial p/\partial\tau = \mu\, \partial p/\partial n$, $|\mu| \ll 1$, and no assumptions are made concerning the relative entropy.

The conditions (3.1), (3.2), with the aid of (1.4), can be written in the form of relations connecting derivatives of ρ and s with respect to u_1, u_2 up to and including the third order. Using the smallness of ε and λ and estimating the terms in system (1.3), we finally obtain from (1.3) the following second-order equation for the density ρ:

$$\gamma p p'(p'\rho - \gamma p \rho_r)(\rho\rho_{\varphi\varphi} - \rho_\varphi^2) + \gamma p \rho' \rho \rho_\varphi(\gamma p r \rho_{r\varphi} - p'\rho_\varphi - \gamma p \rho_\varphi / r) + r^2\rho^2(p'\rho - \gamma p \rho'_r) = 0. \tag{3.3}$$

Here $u_1 = r\cos\varphi$, $u_2 = r\sin\varphi$, $p(r)$ is an arbitrary function of r, and a prime denotes differentiation with respect to r. Both equations of system (1.3) lead to this equation if, for the pressure p, the relation $p_1 u_2 - p_2 u_1 = 0$ ($p_i = \partial p/\partial u_i$), obtained from a linear combination of the original equations, is satisfied. The flow in the ξ_1, ξ_2 plane is determined by the relations

$$\xi_1 = u_1 - \frac{\gamma p p' \rho_\varphi u_2}{\rho r^2(p'\rho - \gamma p \rho_r)}, \qquad \xi_2 = u_2 + \frac{\gamma p p' \rho_\varphi u_1}{\rho r^2(p'\rho - \gamma p \rho_\varphi)}. \tag{3.4}$$

We indicate some particular solutions of (3.3) that may be used in solving concrete problems.

Putting $p = Ar^{k+2}$ and $\rho = r^k\Phi(\varphi)$, for $\Phi(\varphi)$ we obtain

$$\varphi = \int \frac{d\Phi}{(\alpha\Phi^3 + B\Phi^{-2\beta})^{1/2}} + C,$$

where $\alpha = (k + 2 - \gamma k)/\gamma C(k + 2)$, $\beta = (2\gamma k - 2(k + 2) - \gamma)/(k + 2 - \gamma k)$; A, B, C, k are arbitrary constants.

Putting $\rho = f(r)/\mu(\varphi + \varphi_0)^2$, $\mu = \text{const}$, $\varphi_0 = \text{const}$, for $f(r)$ we obtain the ordinary equation

$$2\gamma p' \left(p' f - \gamma p f'\right) - 4\gamma p p' \left(p' f + \frac{\gamma p f}{r} - \gamma p f'\right) + \frac{r^2}{\mu} \left(p' f - \gamma p f'\right)^2 = 0,$$

the function $p(r)$ here remains arbitrary.

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Note: Figure translations are in progress. See original paper for figures.