THE CATEGORY OF HARISH-CHANDRA MODULES OVER THE LIE ALGEBRA OF THE LORENTZ GROUP

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 519.46

MATHEMATICS

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In essence, this work investigates irreducible infinitesimal representations of the Lorentz group. The main result of the article is Theorem 4, which shows that in a special case a series of new, completely nontrivial invariants of representations arises. The authors have succeeded in reducing the problem to a purely algebraic one by introducing a certain special class of infinite-dimensional modules, which we have called Harish-Chandra modules.

Let L be a semisimple Lie algebra over the field of real numbers. Denote by L_k the subalgebra corresponding to some maximal compact subgroup. Consider a module M over the Lie algebra.

Definition. A module M is called a **Harish-Chandra module** over the Lie algebra L if, when regarded as a module over L_k, it is a direct sum $\oplus_i M_i$ of submodules M_i. Here M_i is an irreducible module over L_k, and for each M_i in the sequence $\{M_i\}$ there exist no more than finitely many equivalent to it.

A Harish-Chandra module is called irreducible if it cannot be decomposed into a direct sum of irreducible modules.

1. In this work the irreducible Harish-Chandra modules over the Lie algebra L of the proper Lorentz group are classified. The maximal compact subgroup in this case is isomorphic to the group of rotations of three-dimensional space. By L_k we shall denote the corresponding subalgebra. (In what follows, the notation L and L_k has only this meaning.) In the Lie algebra L one can choose a basis $(h_+, h_-, h_3, f_+, f_-, f_3)$ of six elements. Here the elements h_+, h_-, h_3 form a basis of the subalgebra L_k. A representation in the space M is determined by the images of the basis elements of the algebra L. They will be denoted by $H_+, H_-, H_3, F_+, F_-, F_3$. Recall that these operators H and F are related by the relations

$$
\begin{gathered}
[H_+, H_3] = -H_+; \quad [H_-, H_3] = H_-; \quad [H_+, H_-] = 2H_3; \\
[F_+, H_+] = [F_-, H_-] = [F_3, H_3] = 0; \\
[H_+, F_3] = [F_+, H_3] = -F_+; \quad [H_-, F_3] = [F_-, H_3] = F_-; \\
[H_+, F_-] = [F_-, H_+] = 2F_3; \quad [F_+, F_3] = H_+; \quad [F_-, F_3] = -H_-; \\
[F_+, F_-] = -2H_3.
\end{gathered} \tag{1}
$$

Let further $U(L)$ be the enveloping algebra of the Lie algebra L; let Z be the center of the enveloping algebra. In the center Z one can choose two generators. The operators corresponding to these generators will be denoted by Δ_1 and Δ_2. They are called Laplace operators and are expressed as follows in terms of the operators H and F:

$$
\Delta_1 = \frac{1}{2}(H_- F_+ + F_- H_+) + H_3 F_3; \qquad \Delta_2 = H_- H_+ - F_- F_+ + H_3^2 - F_3^2 + 2H_3. \tag{2}
$$

Theorem 1. *A Harish-Chandra module over the Lie algebra of the proper Lorentz group is decomposable into a direct sum of a countable number of irreducible modules. In each irreducible module the Laplace operators Δ_1 and Δ_2 have one eigenvalue each, λ_1 and λ_2, respectively.*

We say that an operator A has one eigenvalue λ in a Harish-Chandra module if there exists such a number k that $(A - \lambda E)^k = 0$.

Assertion 1. Let M and M' be two Harish-Chandra modules over the Lie algebra L, and suppose that in each of them the Laplace operators have one eigenvalue each, λ_1, λ_2 and λ_1', λ_2', respectively. $\mathrm{Hom}(M, M') \neq 0$ if and only if $\lambda_1 = \lambda_1'$ and $\lambda_2 = \lambda_2'$.

Thus, the study of the category of Harish-Chandra modules over the algebra L may be reduced to the study of a category of modules in each of which the Laplace operators have one eigenvalue each. We shall denote this category by $C(\lambda_1, \lambda_2)$.

Assertion 2. Let $M \in C(\lambda_1, \lambda_2)$, and let M be an indecomposable module. Then there exists a nonnegative number l_0, integral or half-integral, and the numbers $\lambda_1, \lambda_2, l_0$ are related by

$$
l_0^4 + (1 + \lambda_2) l_0^2 - \lambda_1^2 = 0. \tag{3}
$$

Let l_1 be a number satisfying the identities

$$
l_1^2 l_0^2 = -\lambda_1^2, \quad l_1^2 + l_0^2 = -1 - \lambda_2. \tag{4}
$$

We shall call the pair of eigenvalues (λ_1, λ_2) special if l_1 is real and the difference $(|l_1| - l_0)$ is a positive integer. The category $C(\lambda_1, \lambda_2)$ with such a pair will be called special. In the opposite case the pair and the category $C(\lambda_1, \lambda_2)$ will be called nonspecial.

2. The nonspecial case. Introduce the category S. Any object of it is a pair (P, a), where P is a finite-dimensional vector space and a is a linear nilpotent mapping $a : P \to P$. The morphisms $\gamma : (P_1, a_1) \to (P_2, a_2)$ are all such linear mappings $\gamma : P_1 \to P_2$ that $a_2\gamma = \gamma a_1$.

Theorem 2. The nonspecial category $C(\lambda_1, \lambda_2)$ is equivalent to the category S.

Thus, indecomposable modules M correspond to indecomposable objects in the category S. An indecomposable object in S is such a pair (P, a) in which the transformation a is represented by a matrix with one Jordan cell. Consequently, an indecomposable nonspecial module is determined by the following invariants: the numbers λ_1 and λ_2, and the integer parameter n, the size of the Jordan cell.

3. The special category $C(\lambda_1, \lambda_2)$. Introduce the category S_0. Any object A of it consists of two arbitrary finite-dimensional spaces P_1 and P_2 over the field F, and any three mappings $d_+ : P_1 \to P_2$, $d_- : P_2 \to P_1$, $\delta : P_2 \to P_2$, which satisfy the condition $d_-\delta = \delta d_+ = 0$ and the requirement that the operators δ and d_+d_- be nilpotent. The morphisms $\Gamma : A_1 \to A_2$ are all such collections of linear mappings (γ_1, γ_2) for which the diagram is commutative

$$\begin{array}{ccccccc} P_{1,1} & \xrightarrow{d_+} & P_{1,2} & \xrightarrow{\delta} & P_{1,2} & \xrightarrow{d_-} & P_{1,1} \\ \gamma_1 \downarrow & & \gamma_2 \downarrow & & \gamma_2 \downarrow & & \gamma_1 \downarrow \\ P_{2,1} & \xrightarrow{d_+} & P_{2,2} & \xrightarrow{\delta} & P_{2,2} & \xrightarrow{d_-} & P_{2,1} \end{array} \tag{5}$$

In what follows we shall assume that the field F is the field of complex numbers; however, the assertion about the category S_0 will not change if F is an arbitrary algebraically closed field of characteristic not equal to zero.

Theorem 3. The special category $C(\lambda_1, \lambda_2)$ is isomorphic to the category S_0.

It follows from Theorem 3 that the indecomposable modules in the special category $C(\lambda_1, \lambda_2)$ are put in one-to-one correspondence with the indecomposable objects of the category S_0, and conversely. Let us give a description of the canonical form of an indecomposable object into a direct sum in the category S_0. Indecomposable objects may be of two types. Objects of the first type are called nonclosed, and of the second—closed. The simplest type of a nonclosed object will be called a ce-

point. If the object A is a chain, then in the space P_1 one can choose a basis of n vectors $e_1, e_2, \dots, e_{n-1}, e_n$, and in the space P_2 a basis of $(n + m + 1)$ vectors $f_0, f_1, \dots, f_n; f'_1, f'_2, \dots, f'_m$. The operators d_+, d_-, δ are then defined as follows:

$$d_-f_i = e_{i+1}\ (i < n); \qquad d_-f_n = 0; \qquad d_-f'_i = 0; \qquad d_+e_i = f_i;$$

$$\delta f_i = 0 \ (i > 0); \qquad \delta f_0 = f'_1; \qquad \delta f'_i = f'_{i+1} \ (i < m); \qquad \delta f'_m = 0.$$

The vectors f_n and f'_m in the chain will be called tail vectors. The invariants of a chain are two numbers (n, m).

An unclosed object is composed of chains and is specified by a set of numbers

$$(s, n_1, m_1; n_2, m_2; n_3, m_3; \dots, n_k, m_k),$$

where s is equal either to 0 or to 1, and a pair of numbers n_i, m_i determines the i-th chain, with $n_1 \geq 0$; $n_i > 0$ $(i \neq 1)$; $m_i > 0$ $(i \neq k)$; $m_k \geq -1$. Here the tail vectors of these chains are connected by the relations

$$f'_{m_1} = f_{n_2}; \qquad f'_{m_2} = f_{n_3}; \dots, f'_{m_{k-1}} = f_{n_k}. \tag{7}$$

If $s = 0$, then these relations completely determine the unclosed object. If $s = 1$, then in the first chain the tail vector f_{n_1} is equal to zero, i.e. in this case $d_+ e_{n_1} = 0$. If $m_k = -1$, then in the k-th chain the vector f_{0k} is equal to zero, i.e. in this case the vector e_{1k} has no preimage with respect to the operator.

Assertion 3. *Let B and B' be two unclosed indecomposable objects, and let them be specified by the sets of numbers $(s, n_1, m_1; n_2, m_2; n_3, m_3; \dots; n_{k-1}, m_{k-1}; n_k, m_k)$ and $(s', n'_1, m'_1; n'_2, m'_2; \dots; n'_{k-1}, m'_{k-1}; n'_k, m'_k)$. The objects B and B' are equivalent (coincide) if and only if $s = s'$, $n'_i = n_i$, $m'_i = m_i$.*

A closed object in the simple case can be obtained from a single unclosed object determined by a set of numbers with $s = 0$, $n_1 > 0$; $m_k > 0$. In this case the vectors f_{n_1} and f'_{m_k} are nonzero. We shall call them the initial and terminal vectors of the unclosed object. The closed simple object is determined if, to the relations (7), one more is added,

$$f'_{m_k} = \mu f_{n_1}, \tag{8}$$

where μ is an arbitrary complex number.

Thus, in the simple case a closed object is specified by a set of numbers

$$(n_1, m_1; n_2, m_2; \dots; n_k, m_k; \mu),$$

where $n_i, m_i > 0$.

In the general case a closed object is specified by a set of numbers

$$(n_1, m_1; n_2, m_2; \dots; n_k, m_k; \mu; N),$$

where $n_i, m_i, N > 0$. This object is composed of unclosed objects, each of which is determined by the same set of numbers

$$(0, n_1, m_1; n_2, m_2; \dots; n_k, m_k).$$

Denote the initial and terminal vectors of the j-th $(j = 1, \dots, N)$ unclosed object by f_j and f'_j. Then the closed object is determined by specifying the following relations between these vectors:

$$f'_1 = \mu f_1; \qquad f'_2 = \mu f_2 + f_1; \dots ; f'_i = \mu f_i + f_{i-1}, \ \dots \ (i = 2, 3, \dots , N). \tag{9}$$

Assertion 4. *Let B and B' be two unclosed indecomposable objects from the category S_0, and let them be specified by the sets of numbers $(n_1, m_1; n_2, m_2; \dots ; n_k, m_k, \mu, N)$ and $(n'_1, m'_1; \dots ; n'_k, m'_k, \mu', N')$. The objects B and B' coincide if and only if $\mu = \mu'$, $N = N'$, and the sequence $\{n'_i, m'_i\}$ is a cyclic permutation of the sequence $\{n_i, m_i\}$.*

Theorem 4. *Every indecomposable object from the category S_0 is either unclosed or closed, and the constructions indicated above exhaust all indecomposable objects.*

Corollary. Let M be an indecomposable Harish-Chandra module over the Lie algebra L, let M belong to the special category $C(\lambda_1, \lambda_2)$, and let the space M be a vector space over the field of complex numbers.

Then there are two alternatives: either the object is described by a set of integers, or the object is described by a set of integers and by the complex number μ. In order that the modules M and M' be equivalent, it is necessary and sufficient that $\lambda_1 = \lambda'_1$, $\lambda_2 = \lambda'_2$, $\mu = \mu'$, and that the corresponding sets of numbers coincide.

It is quite remarkable that an indecomposable module from a special category can have a complex number as an invariant.

In the work of D. P. Zhelobenko [2], a representation of the Lorentz group was considered, called a representation of finite rank. There the representation which, in our terminology, is called special is completely analyzed. However, in the special case there, the problem is only reduced to a certain algebraic one. The system of invariants of an indecomposable special object and, in particular, the presence of the continuous parameter μ and the canonical form are absent there.

Received
6 VI 1967

CITED LITERATURE

[1] I. M. Gel' fand, R. A. Minlos, Z. Ya. Shapiro, *Representations of the Rotation Group and of the Lorentz Group*, Moscow, 1958.
[2] D. P. Zhelobenko, DAN, **126**, No. 5, 935 (1959).

Note: Figure translations are in progress. See original paper for figures.