CERTAIN PROPERTIES OF AN EXTREMAL SEQUENCE OF REGULARLY MONOTONE POLYNOMIALS

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Abstract

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CERTAIN PROPERTIES OF AN EXTREMAL SEQUENCE OF REGULARLY MONOTONE POLYNOMIALS

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1°. We shall consider the class ${}_{n}^{(\bar{\lambda}_s)}$ of regularly monotone polynomials $P_n(x)$, $P_n^n(x) \equiv 1$, of order n on $[0,1]$ with type numbers $\mu_1, \mu_2, \dots, \mu_m$, where

$$\sum_{j=1}^{m} \mu_j = n, \qquad \mu_{ps+l} = \lambda_l, \quad l = 1, \dots, s, \quad \mu_m \leq \lambda_{l_0},$$

$m = p_0 s + l_0$. Let, for definiteness, the first type number $\mu_1 = \lambda_1$ correspond to a permanence. From the generalized theorem of S. N. Bernstein (([2], p. 515)) it follows that in the class ${}_{n}^{(\bar{\lambda}_s)}$ the polynomial $P_n(x)$ (hereafter—the extremal polynomial) deviates least from zero on $[0,1]$, and together with its derivatives satisfies the conditions $P_n^{(\nu)}(\alpha_\nu) = 0$, where $\alpha_\nu = 0$, if

$$\begin{aligned} \nu - qb_s &= \overline{b_{2k},\, b_{2k+1}^{-1}}, && s = 2p, && k = 0, 1, \dots, (s-2)/2, \\ \nu - 2qb_s &= \overline{b_{2k},\, b_{2k+1}^{-1}}, && s = 2p+1, && k = 0, 1, \dots, (s-1)/2, \\ \nu - (2q+1)b_s &= \overline{b_{2k+1},\, b_{2k+2}^{-1}}, && s = 2p+1, && k = 0, 1, \dots, (s-3)/2, \end{aligned} \tag{A}$$

and $\alpha_\nu = 1$, if

$$\begin{aligned} \nu - qb_s &= \overline{b_{2k+1},\, b_{2k+2}^{-1}}, && s = 2p, && k = 0, 1, \dots, (s-2)/2, \\ \nu - 2qb_s &= \overline{b_{2k+1},\, b_{2k+2}^{-1}}, && s = 2p+1, && k = 0, 1, \dots, (s-3)/2, \\ \nu - (2q+1)b_s &= \overline{b_{2k},\, b_{2k+1}^{-1}}, && s = 2p+1, && k = 0, 1, \dots, (s-1)/2, \end{aligned} \tag{B}$$

where

$$b_p = \sum_{j=1}^{p} \mu_j, \qquad b_0 = 0.$$

(We note that for the class under consideration

$$b_{qs+i} = qb_s + b_i, \quad i = 0, \dots, s-1, \quad q = 0, 1, 2, \dots .)$$

In what follows we shall write $\nu \in A$, if ν satisfies conditions (A), and $\nu \in B$, if ν satisfies conditions (B). Let r be an integer nonnegative number. Consider the sequence $\{P_{n,r}(x)\}_{n=0}^{\infty}$, where

$$P_{n,r}(x) \equiv P_{n+r}^{(r)}(x),$$

and the polynomial $P_{n+r}(x) \in {}^{(\bar\lambda_s)}_{n+r}$ is extremal in this class. Represent r in the form $r = qb_s + b_i + j$ $(i = 0, \dots, s-1,\ j = 0, \dots, \lambda_{i+1} - 1)$. It is not difficult to see that any polynomial $P_{n,r}(x)$ of the sequence under consideration is a regularly monotone polynomial of order n on $[0, 1]$ with type numbers

$$\lambda_{i+1} - j, \lambda_{i+2}, \dots, \lambda_s, \lambda_1, \lambda_2, \dots, \lambda_s, \lambda_1, \dots \tag{1}$$

The class of regularly monotone polynomials of order n on $[0, 1]$ with type numbers (1) will be denoted by ${}^{(\bar\lambda_s)}_{n,r}$.

Theorem 1. Of all polynomials $y_{n,r}(x) \in {}^{(\bar\lambda_s)}_{n,r}$, $y_{n,r}^{(n)}(x) \equiv 1$, the polynomial $P_{n,r}(x)$ deviates least from zero on $[0, 1]$, for which the conditions

$$P_{n,r}^{(\nu)}(\alpha_\nu^{(r)}) = 0, \qquad \nu = 0, \dots, n-1,$$

are satisfied, where

$$\alpha_\nu^{(r)} = \begin{cases} 0, & \text{if } \nu + r \in A, \\ 1, & \text{if } \nu + r \in B. \end{cases} \tag{2}$$

Moreover, the magnitude of the least deviation is

$$L_n = \left|P_{n,r}(1 - \alpha_0^{(r)})\right|.$$

Theorem 2. The sequence $\{P_{n,r}(x)\}_{n=0}^{\infty}$ of extremal polynomials of the class ${}^{(\bar\lambda_s)}_{n,r}$ is a sequence of generalized Appell polynomials for the class $A_k^{(\bar\lambda_s)}$ (see (3)). Here $k = b_s$, if s is even, and $k = 2b_s$, if s is odd.

Theorem 3. Every polynomial $P_{n,r}(x)$ of the extremal sequence $\{P_{n,r}(x)\}_{n=0}^{\infty}$ of the class ${}^{(\bar\lambda_s)}_{n,r}$ can be represented in the form

$$P_{n,r}(x) = \frac{1}{n!} \sum_{\nu=0}^{n} C_n^\nu E_{n-\nu}^{n,r} x^\nu,$$

where the numbers $E_\nu^{n,r}$ are successively determined from the system of equations

$$\begin{aligned} E_0^{n,r} &= 1, \\ E_{n-\nu}^{n,r} &= 0, \quad \text{if } \nu + r \in A, \\ (1 + E^{n,r})_{n-\nu} &= 0, \quad \text{if } \nu + r \in B. \end{aligned}$$

2°. Denote by ${}^{(\bar\lambda_s)}_{n-l,r}$, $l < n-1$, the class of regularly monotone polynomials $y_{n,r}(x)$ of order $n - l$ on $[0, 1]$ with type numbers (1).

Theorem 4. Of all polynomials $y_{n,r}(x) \in {}^{(\bar{\lambda}_s)}_{n-l,r}$ of the form

$$y_{n,r}(x) = \sum_{k=0}^{n} \sigma_k x^k,$$

where the coefficients σ_k, $k = n-l, \dots, n$, are fixed, the polynomial

$$y^*_{n,r}(x) = \sum_{k=n-l}^{n} a_k P_{k,r}(x),$$

deviates least from zero on $[0,1]$, with

$$a_k = \begin{cases} \sigma_k k!, & \text{if } k+r \in A, \\ \displaystyle\sum_{m=k}^{n} \frac{m!}{(m-k)!}\,\sigma_m, & \text{if } k+r \in B. \end{cases}$$

Denote by $\mathfrak{P}$ the subclass of polynomials

$$y_{n,r}(x) = \sum_{k=n-l}^{n} \sigma_k x^k + \sum_{k=0}^{n-l-1} P_k x^k \in {}^{(\bar{\lambda}_s)}_{n-l,r}$$

with m, $0 \le m \le l$, fixed coefficients $\sigma_{k_1}, \sigma_{k_2}, \dots, \sigma_{k_m}$,

$$n-l \le k_1 < k_2 < \dots < k_m \le n.$$

Theorem 5. If there exists a polynomial $y^*_{n,r}(x) \in \mathfrak{P}$ such that for any polynomial $y_{n,r}(x) \in \mathfrak{P}$ and any $x \in [0,1]$ the inequality

$$\left|y^{*(n-l)}_{n,r}\right| \le \left|y^{(n-l)}_{n,r}(x)\right|$$

is satisfied, then it is the polynomial least deviating from zero on $[0,1]$.

In particular, the following holds.

Theorem 6. Among all polynomials $y_{n,r}(x) \in {}^{(\bar{\lambda}_s)}_{n-2,r}$ with fixed leading coefficient σ_n such that

$$y^{(n-2)}_{n,r}(x)\sigma_n \le 0, \qquad x \in [0,1],$$

the polynomial

$$y^*_{n,r}(x) = \begin{cases} n!\,\sigma_n\left[P_{n,r}(x) - \frac{1}{2}P_{n-1,r}(x)\right], & \text{if } n-1+r \in A, \\ n!\,\sigma_n\left[P_{n,r}(x) + \frac{1}{2}P_{n-1,r}(x)\right], & \text{if } n-1+r \in B. \end{cases}$$

deviates least from zero on $[0,1]$.

3°. Consider the sequence $\{P_{n,r}(x)\}_{n=0}^{\infty}$ of extremal polynomials of the class ${}^{(\bar{\lambda}_s)}_{n,r}$, satisfying the conditions of Theorem 1. Let L be a linear differential operator generated by the differential expression $l(f) = f^{(\tilde{k})}$ and by the separated boundary conditions $f^{(\nu)}(\alpha_\nu^{(r)}) = 0,\ \nu = 0, \dots, \tilde{k}-1$. If $\bar{\lambda}_s$ and r are such that the corresponding boundary-value problem has a unique eigenvalue ρ_1 of smallest modulus (see (4)), then the following holds.

Theorem 7. *If for L one of the following conditions is fulfilled: 1) $L = L^*$, 2) $L \neq L^*$, but all eigenvalues of the operator L are simple zeros of the characteristic determinant $\Delta(\rho)$, then the asymptotic equality holds*

$$\lim_{n\to\infty} \frac{P_{n,r}(x)}{P_{n,r}(1-\alpha_0^{(r)})} = \frac{\varphi_1(x)}{\varphi_1(1-\alpha_0^{(r)})},$$

where $\varphi_1(x)$ is the eigenfunction of the operator L corresponding to the eigenvalue ρ_1 of smallest modulus.

From this theorem there immediately follows a refinement of one result of Tagamlitskii (see [5], p. 199):

Theorem 8. *Let $f(x)$ be a regularly monotone function with an infinite order of monotonicity and with type numbers (1). Then it can be represented in the form*

$$f(x) = \sum_{k=0}^{\infty} c_k P_{k,r}(x) + A\,\frac{\varphi_1(x)}{\varphi_1(1-\alpha_0^{(r)})},$$

where c_k and A are certain nonnegative constants.

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Note: Figure translations are in progress. See original paper for figures.