A GENERALIZATION OF A THEOREM OF A. N. KOLMOGOROV ON POINTS OF LOCAL TOPOLOGICITY AND ITS CONSEQUENCES

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Abstract

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(Presented by Academician P. S. Aleksandrov on 4 IV 1966)

In the paper ([1]) the following theorem was proved:

If there is an open countable-to-one mapping f of a compactum X, then the set T_f of points of local topologicity of the mapping f is dense in X.*

Here we give a generalization of this theorem which makes it possible to assert something new already for spaces with a countable base. In addition, analogues of this generalization are given here for the case of finite-to-one mappings.

Definition. A mapping $f : X \to Y$ is called **preopen** if, whenever $fU \cap fV \neq \Lambda$, where U and V are open in X, it follows immediately that $\langle fU \rangle \cap \langle fV \rangle \neq \Lambda$. Here $\langle R \rangle$ denotes the set of interior points of R.

Preopen mappings include the class of semiopen** mappings and include the class of open mappings.

Here and below, by a space we mean a Hausdorff space.

Theorem 1. *Let there be a locally countable-to-one preopen mapping $f : X \to Y$, where X is a locally bicompact space. Then the set T_f of points of local topologicity of the mapping f is dense in X.*

Proof. Suppose, on the contrary, that there exists an open set $U \subset X$ with bicompact closure, containing no points of local topologicity, and such that f is countable-to-one on U. On the basis of this assumption, we shall find a point $y^\# \in \langle fU \rangle = V$ such that the set $f^{-1}y^\# \cap U$ is uncountable, which will constitute a contradiction.

Construct a system of open sets and points

$$\left.\begin{array}{l} x^{(n)}_{i_1 \dots i_n} \in U \\ y^{(n)} \in V \\ U^{(n)}_{i_1 \dots i_n} \subset U \\ V^{(n)} \subset V \end{array}\right\} \quad n = 1, 2, \dots ;\ i_k = 0, 1,$$

with the following properties:

$$
\begin{aligned}
&1)\quad f x^{(n)}_{i_1\dots i_n} = y^{(n)},\\
&2)\quad x^{(n)}_{i_1\dots i_n} \in U^{(n)}_{i_1\dots i_n},\\
&3)\quad y^{(n)} \in [V^{(n)}],\\
&4)\quad [V^{(n+1)}] \subset [V^{(n)}],\\
&5)\quad V^{(n)} \subset \langle f U^{(n)}_{i_1\dots i_n}\rangle,\\
&6)\quad [U^{(n+1)}_{i_1\dots i_n i_{n+1}}] \subset U^{(n)}_{i_1\dots i_n},\\
&7)\quad [U^{(n+1)}_{i_1\dots i_n 0}] \cap [U^{(n+1)}_{i_1\dots i_n 1}] = \Lambda.
\end{aligned}
$$

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* Earlier, for the case of finite-to-one mappings, a similar theorem was proved by P. S. Aleksandrov ([2]).

** A mapping $f : X \to Y$ is semiopen if $\langle fU\rangle$ is nonempty for every open $U \subset X$.

Suppose that such a system has been constructed. Let us show how everything will follow from this. For any sequence

$$i_1, i_2, \dots, i_n, \dots \tag{1}$$

the set of points $\{x^{(n)}_{i_1\dots i_n}\} = A_{i_1\dots i_n\dots}$ is such that $f x^{(n)}_{i_1\dots i_n} = y^{(n)}$; $\{y^{(n)}\} = B$; $fA_{i_1\dots i_n\dots} = B$. Denote by $y^\#$ an arbitrary point of complete accumulation of the countable set B, $y^\# \in [B]$. There exists a point $x_{i_1\dots i_n\dots}$, which is a point of complete accumulation for $A_{i_1\dots i_n\dots}$, such that

$$f x_{i_1\dots i_n\dots} = y^\#; \qquad x_{i_1\dots i_n\dots} \in \bigcap_{n=1}^{\infty} [U^{(n)}_{i_1\dots i_n}].$$

Such a point $x_{i_1\dots i_n\dots}$ can be found starting from any sequence of the form (1), and from condition 7) it follows that for two different such sequences these points are distinct. Hence $f^{-1}y^\# \cap U$ is uncountable, and the required contradiction has been obtained.

Let us outline the construction of the system of points and sets possessing properties 1)–7). As $y^{(1)}$ we take any point of $V = \langle fU\rangle$ which has at least two points in its inverse image lying in U; denote them by x_0, x_1. As $U^{(1)}_0$ and $U^{(1)}_1$ we take neighborhoods of the points x_0 and x_1 such that

$$[U^{(1)}_0] \cup [U^{(1)}_1] \subset U; \qquad [U^{(1)}_0] \cap [U^{(1)}_1] = \Lambda.$$

As $V^{(1)}$ we take an arbitrary open set such that

$$y^{(1)} \in [V^{(1)}], \qquad V^{(1)} \subset V \cap \langle fU_0^{(1)} \rangle \cap \langle fU_1^{(1)} \rangle,$$

which is possible by virtue of the preopenness of f. The further construction is carried out by double induction according to Kolmogorov' s method (with the necessary modifications).

In the theorem, local bicompactness cannot be replaced by completeness in the sense of Čech.* This follows at least from the fact that the space of irrational points of the interval $[0, 1]$ can be condensed by a preopen mapping without points of local topologicity onto a certain compactum. However, the following theorem is true:

Theorem $1'$**.** *Let there be a preopen locally countably multiple mapping* $f : X \to Y$*, where* X *is either a locally complete space in the sense of Čech or a locally countably compact space. Then the set of points of local condensation of the mapping* f *is dense in* X*.*

A point $x \in X$ is called a **point of local condensation of the mapping** $f : X \to Y$ if it has a neighborhood Ox on which f is a condensation.

Corollary 1. *Let there be a countably multiple preopen mapping* $f : X \to Y$*, where* X *is either a locally complete space in the sense of Čech or a locally countably compact space. Then the* π*-weight of* X *is equal to the* π*-weight of* Y *(see the definition in (3)).*

The corollary is derived from Theorem 1 by means of the following lemma.

Lemma. *Let there be a half-open countably multiple mapping* $f : Z \to W$ *such that every point* $z \in Z$ *is a point of local condensation. Then the* π*-weight of* Z *is equal to the* π*-weight of* W*.*

If $\{\omega_\alpha\}$ is a system of open sets dense in Z, then $\{\langle f\omega_\alpha \rangle\}$ is a dense system of open sets in W, whence it is clear that the π-weight of Z is $\geq$ the π-weight of W. Let us prove the reverse inequality. Let $\omega = \{\omega_\alpha\}$ be a dense system in W; in each ω_α fix an arbitrary point $y_\alpha \in \omega_\alpha$; $f^{-1}y_\alpha = \{x_{\alpha i}\}$. For every point $x_{\alpha i}$ take a neighborhood $Ox_{\alpha i}$ such that $fOx_{\alpha i} \subseteq \omega_\alpha$, and such that f is a condensation on $Ox_{\alpha i}$. All possible pairwise intersections of elements of the system $\{Ox_{\alpha i}\}$ give us a dense system whose cardinality is equal to the cardinality of the system ω.

* However, for the case of open mappings this is possible, as B. A. Pasynkov has recently shown.

Corollary 2. *Suppose there is a countably multiple preopen mapping* $f : X \to Y$*, where* X *is a topologically homogeneous bicompactum (or even homogeneous with respect to local weight). Then the weight of* X *is equal to the weight of* Y*.*

In connection with the last assertions, let us note that the question of whether the weight of a bicompactum can decrease under a countably multiple open mapping remains unresolved.

Theorem 2. *Suppose there is an open finite-to-one mapping $f : X \to Y$, where Y is a Čech-complete space. Then in X there is a dense subset of points of local topologicality of the mapping f.*

Lemma 1. *A Čech-complete space cannot be represented as the sum of a countable number of nowhere dense subspaces.*

The lemma is proved on the basis of the completeness criterion from (4).

Lemma 2. *An open k-to-one mapping is a locally topological mapping.*

We outline the proof of the theorem. The space

$$Y = \bigcup_{i=1}^{\infty} Y_i,$$

where Y_i is the set of all points of multiplicity i in Y;

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where $X_i = f^{-1} Y_i$. We shall prove that in an arbitrary open set $U \subset X$ there is a point of local topologicality. Since $V = fU$ cannot, by Lemma 1, be represented as the sum of a countable number of nowhere dense sets, there exists an i such that $Y_i \cap V$ is not nowhere dense in V. For every k, the set

$$\bigcup_{j=1}^{k} Y_j$$

is closed in Y, and therefore there exists a set $Q \subset Y_i \cap V$ that is open in Y. Hence, by virtue of the validity of Lemma 2, we conclude that in U there is a point of local topologicality.

Remark. In the hypothesis of Theorem 2, instead of completeness one may require less of Y. It is sufficient to require that Y cannot be represented as the sum of a countable number of nowhere dense subspaces, and the theorem will not lose its force. In particular, if Y is either a locally Čech-complete space or a locally countably compact space, then the theorem remains valid. Moreover, it is sufficient to require from f local finite multiplicity instead of finite multiplicity.

Theorem $2'$. *Suppose there is a boundedly multiple open mapping $f : X \to Y$. Then the set T_f of points of local topologicality of the mapping f is dense in the space X.*

Corollary. *If a dyadic space X is boundedly multiply and openly mapped onto a space with the first axiom of countability, then X has a countable base.*

But there exists a dyadic space that maps finitely and openly onto the rational points of the interval.

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Note: Figure translations are in progress. See original paper for figures.