PARTIAL SUFFICIENCY AND UNBIASED ESTIMATION OF POLYNOMIALS OF A SHIFT PARAMETER

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Abstract

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PARTIAL SUFFICIENCY AND UNBIASED ESTIMATION OF POLYNOMIALS OF A SHIFT PARAMETER

(Presented by Academician Yu. V. Linnik on 26 VII 1966)

1. Let $\{P_\theta;\ \theta \in \Theta\}$ be a family of distributions on the space $(X, \mathfrak{A})$, depending on an abstract parameter $\theta \in \Theta$. A statistic $T(x)$ is called **sufficient for the family** $\{P_\theta\}$ if, for every bounded function $\varphi(x)$,

$$E_\theta(\varphi \mid T) = \widetilde{\varphi} \text{ a.s. } P_\theta, \qquad \theta \in \Theta, \tag{1}$$

for some $\widetilde{\varphi} = \widetilde{\varphi}(T(x))$. The answer to the question of when the statistic $T(x)$ is sufficient for the family $\{P_\theta\}$ is given by the factorization theorem (^{1}). The fundamental role of sufficient statistics in estimation theory is determined by the Rao—Blackwell—Kolmogorov theorem ($^{2-4}$).

Of greatest interest is the case when the family $\{P_\theta\}$ of distributions in R^n is generated by a repeated sample $(x_1, \dots, x_n)$ from a one-dimensional population with distribution function (d.f.) $F(x;\theta)$,

$$P_\theta(A) = \int_{(A)} \cdots \int dF(x_1;\theta) \cdots dF(x_n;\theta). \tag{2}$$

For such families, necessary and sufficient conditions for the existence of nontrivial sufficient statistics, which are naturally formulated in the language of the one-dimensional d.f. $F(x;\theta)$, can be found in (^{5}).

2. Yu. V. Linnik proposed the following definition of partial sufficiency of a statistic $T(x)$. Let $\mathcal{L} = \{\varphi\}$ be some linear system of functions on $(X, \mathfrak{A})$ for which $E_\theta|\varphi| < \infty,\ \theta \in \Theta$. A statistic $T(x)$ is called **$\mathcal{L}$-sufficient** if condition (1) is fulfilled for every $\varphi \in \mathcal{L}$. The analogue of the Rao—Blackwell—Kolmogorov theorem has the following form.

If, for $\varphi \in \mathcal{L}$, $E_\theta\varphi^2 < \infty,\ \theta \in \Theta$, and $T(x)$ is an $\mathcal{L}$-sufficient statistic, then for the function $\widetilde{\varphi} = E_\theta(\varphi \mid T)$ we have

$$E_\theta \widetilde{\varphi} = E_\theta \varphi, \qquad E_\theta(\widetilde{\varphi} - E_\theta \widetilde{\varphi})^2 \leq E_\theta(\varphi - E_\theta \varphi)^2, \qquad \theta \in \Theta.$$

In other words, all estimates from $\mathcal{L}$ are inadmissible, except, perhaps, those that depend only on the $\mathcal{L}$-sufficient statistic.

Let us note that if P_θ has the form (2) and $\int x^{2k}\, dF(x;\theta) < \infty$, $\theta \in \Theta$, then the system $\mathcal{L}$ for which it is natural to seek partially sufficient statistics is the collection of all polynomials in $x_1, \dots, x_n$ of degree not exceeding k.

3. In what follows we shall assume that $(x_1, \dots, x_n)$ is a repeated sample from a population with d.f. $F(x - \theta)$, depending on a shift parameter $\theta \in R^1$, and

$$P_\theta(A) = \int_A \cdots \int dF(x_1 - \theta) \cdots dF(x_n - \theta).$$

It is well known (5,6) that if $F(x)$ is absolutely continuous with respect to Lebesgue measure, and the statistic

$$\bar{x} = \frac{1}{n}(x_1 + \dots + x_n), \qquad n \geq 2,$$

is sufficient for the family $\{P_\theta\}$, then $F(x)$ is the distribution function of the normal law. As the result of note (7) shows, it is not possible to enlarge the family of normal distributions by replacing the sufficiency of the statistic $\bar{x}$ by its partial sufficiency in the sense of item 2. We shall modify the concept of partial sufficiency in the spirit of (8).

Suppose that

$$\int x^{2k}\, dF(x) < \infty \tag{3}$$

for some integer $k \geq 0$. When condition (3) is satisfied, the totality of all polynomials $\pi(x_1, \dots, x_n)$ of degree not higher than k forms a Hilbert space $L_k^{(2)}$, if the inner product of elements π_1 and π_2 is introduced in the usual way:

$$(\pi_1, \pi_2)_\theta = E_\theta(\pi_1 \pi_2).$$

By T_k we shall denote the subspace of L_k^2 generated by the functions

$$a_0 \bar{x}^k + \dots + a_k.$$

We shall say that T_k is an L_k^2-sufficient subspace if, for any $\pi \in L_k^2$,

$$\hat{E}_\theta(\pi \mid T_k) = \tilde{\pi} \tag{4}$$

for some $\tilde{\pi} \in \tilde{T}_k$, where $\hat{E}_\theta(\cdot \mid T_k)$ is the projection operator onto T_k, when the inner product $(\cdot,\cdot)_\theta$ is introduced by means of the measure P_θ.

Theorem 1. *If the first $2k$ moments of the distribution function $F(x)$ coincide with the corresponding moments of some normal law, then T_k is an $L_k^{(2)}$-sufficient subspace.*

Proof. Let first $F(x)$ be the distribution function of a normal law, and let $\pi(x_1, \dots, x_n) = \pi \in L_k^{(2)}$. Obviously, π can be represented in the form

$$\pi = \bar{x}^k q_0(x_2 - x_1, \dots, x_n - x_1) + \dots + q_k(x_2 - x_1, \dots, x_n - x_1),$$

where $q_j(y_2, \dots, y_n)$, $j = 0, \dots, k$, are also polynomials in $y_2, \dots, y_n$. But if $x_1, \dots, x_n$ are normally distributed random variables, then $\bar{x}$ and the vector $(x_2 - x_1, \dots, x_n - x_1)$ are independent. Therefore

$$E_\theta(\pi \mid \bar{x}) = a_0 \bar{x}^k + \dots + a_k, \tag{5}$$

where

$$a_j = E_\theta(q_j \mid \bar{x}) = E_\theta q_j = E_0 q_j. \tag{6}$$

Since $E_\theta(\pi \mid \bar{x}) \in T_k$, it follows that $\hat{E}_\theta(\pi \mid T_k) = E_\theta(\pi \mid \bar{x})$, and, according to (5), T_k is an $L_k^{(2)}$-sufficient subspace if only $F(x)$ is the distribution function of a normal law. But two distributions whose first $2k$ moments are identical induce one and the same inner product in $L_k^{(2)}$. Therefore, for a distribution function $F(x)$ satisfying the condition of Theorem 1, we shall have

$$\hat{E}_\theta(\pi \mid T_k) = a_0 \bar{x}^k + \dots + a_k,$$

where $a_0, \dots, a_k$ are the same as in (6). Theorem 1 is proved.

4. The polynomial $\pi(x_1, \dots, x_n)$ is an unbiased estimate of the function

$$\bar{\pi}(\theta) = E_\theta \pi = c_0 \theta^k + \dots + c_k$$

with finite, if condition (3) is satisfied, variance for all $\theta \in R^1$. Let the quality measure of estimates be their variance. Then from Theorem 1 we obtain the following analogue of the Rao-Blackwell-Kolmogorov theorem.

Theorem 2. *If the first $2k$ moments of the distribution function $F(x)$ coincide with the corresponding moments of some normal law, then every polynomial $\pi(x_1, \dots, x_n) = \pi \in L_k^{(2)} \setminus T_k$ is inadmissible in the class of unbiased estimates of the function $\bar{\pi}(\theta)$.*

The question of whether estimators $\pi \in T_k$ are admissible or not requires special study. However, it is the optimal unbiased estimator of the function $\pi(\theta)$ for all $\theta \in R^1$ in one single case, as the following result shows.

Theorem 3. *Let the distribution function $F(x)$ satisfy condition (3), and let the polynomial*

$$\pi(\bar{x}) = a_0 \bar{x}^k + \dots + a_k, \quad a_0 \neq 0,$$

of degree $k \geq 1$*, for some* $n \geq 3$*, be the best unbiased estimator, for all* $\theta \in R^1$*, of the function*

$$\pi(\theta) = c_0\theta^k + \cdots + c_k.$$

Then $F(x)$ *is the distribution function of the normal law.*

As shown in (9), for $k = 1$ Theorem 3 remains valid if one assumes only the admissibility of the estimator $a_0\bar{x} + a_1$.

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Note: Figure translations are in progress. See original paper for figures.

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