Incorrect problems with a closed non-invertible operator

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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ILL-POSED PROBLEMS WITH CLOSED NON-INVERTIBLE OPERATORS

An element $x' \in M$ is called a quasi-solution of equation (1) on the set M (or on a compact set M) if:

$$\rho(Ax', y) = \min_{x \in M} \rho(Ax, y).$$

In other words, a quasi-solution is the preimage of some projection of the point y onto the set AM. The (complete) set of quasi-solutions is, evidently, the complete preimage in M of the set of projections $Ty = A^{-1}Py = A^{-1}\Phi y = Ty$. Theorem 2 from work [?] is erroneous, as demonstrated by counterexamples. Theorem 1 of the present section confirms the validity of all other results in [?], except for the final conclusion of Theorem 5.

From this, it follows, among other things, that the set of quasi-solutions is non-empty when M is a compact set and coincides with the set of exact solutions of equation (1) if such solutions exist—that is, if $y \in AM$.

Theorem. A non-empty metric projection P of a metric space Y onto a subset AM is β-continuous at every point $y \in U$.

Proof. Take $y \in U$ and $y_\delta \in U$ such that $\rho(y, y_\delta) < \epsilon/2$. Since $Py \subset AM$, we have $\rho(y, Py) = \rho < \epsilon/2$. According to (3), $Py_\delta = N \cap S(y, \delta) \subset S(y, \epsilon) \subset U(y, \epsilon) = U(Py, \epsilon)$, which was to be proved. For this theorem to hold, it is evidently sufficient that there exists a neighborhood of the element y, every point of which has a (non-empty) projection in AM. If M is a compact set, the mapping P is β-continuous on the entire space (see [**?**]).

Definition.## Ill-Posed Problems with a Closed Irreversible Operator

In the study of ill-posed problems involving a closed irreversible operator [**?**], the requirement that the right-hand side of equation (1) belongs to the image of the aforementioned bicompact set becomes non-essential. However, the requirements for the continuity and one-to-one correspondence of the operator are typically maintained.

Similar circumstances arise in a different approach developed by V. K. Ivanov. Here, the solution is still sought on a bicompact set; however, since the data may lie outside its image in F, it is proposed to construct the best approximation to the data within this image and take the preimage of that best approximation. The "quasisolution" constructed in this manner (see Definition 5 below) generalizes the exact solution and, in some cases, even proves to be well-posed in the sense of Hadamard (see, for example, [**?**]).

The strictly geometric nature of this new approach has opened possibilities for further generalizations. Among these, we highlight the ability to weaken the requirements on the operator A, specifically by abandoning the requirement of its invertibility [**?**]. In this case, while stability of the usual type generally loses its meaning, a certain generalized stability—so-called β-continuity (see Definition 3 below)—is preserved for a continuous operator.

Finally, very recently, V. K. Ivanov [**?**] conversely maintained the one-to-one correspondence of the linear operator but abandoned its continuity, replacing it with closedness in the sense of Definition 1. The stability of the quasisolution is not lost in this case either, provided that F is a Banach E-space. The transition to closed operators allows this theory to be applied directly to a wide range of differential problems.

In the present work, we also restrict ourselves to the property of operator closedness, which is weaker than continuity, but simultaneously abandon the requirement of operator invertibility. Although this case differs significantly from that considered in [**?**] (the image N of a relatively compact set from D possesses only closedness rather than compactness as in [**?**]), it is nonetheless possible to prove the β-continuity of the inverse (multivalued) mapping $A^{-1} : N \to M$ (Section 4). Furthermore, if metric projections onto N in the sense of Definition 4 exist, we can demonstrate the β-continuity of the projection operation and, consequently, of the quasisolutions themselves at points in N (Section 5).

Subsequently, in Sections 6 and 7, we establish certain criteria for the uniqueness and existence of metric projections onto a closed convex set, which are utilized for the theory of quasisolutions of equation (1) with a linear operator (Sections 7 and 8). By narrowing the class of spaces to E-spaces, the condition $y_0 \in N$ is removed, as the projection and quasisolutions become β-continuous on the entire E-space (Section 9). Furthermore, in Section 10, the β-stability of quasisolutions is estimated via the modulus of β-continuity of the mapping A^{-1} at a point in N for a Hilbert space F. For this same case, Section 11 outlines a method for the approximate determination of quasisolutions on M, consisting of their construction on an expanding sequence of subsets whose union is dense in a specific manner.

§ 1. STABILITY

Quasisolutions

Let X and Y be metric spaces, and let A be an operator with domain $D(A) \subseteq X$ and range $R(A) \subseteq Y$. Given an element $y \in Y$, the objective is to determine an element $x \in X$ such that the following equality holds:

$$Ax = y.$$

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Since the element y does not necessarily belong to the range $R(A)$, and the inverse operator A^{-1} is assumed to be neither continuous nor even necessarily existent (i.e., single-valued), any of Hadamard' s conditions for well-posedness may be violated.

Definition.

1. Operator

An operator $A : X \to Y$ with domain $D(A)$ is called closed if its graph is closed in $X \times Y$; that is, if the relations $x_n \to x$, $Ax_n \to y$ (as $n \to \infty$) imply that $Ax = y$ (see [**?**, p. 70]). We shall assume the operator A in (1) is closed. Let $\mathfrak{M} \subset X$ be a compact set (in metric spaces, the concepts of compact and bicompact coincide), and let $M = D(A) \cap \mathfrak{M}$.

We will show that the (generally multi-valued) inverse mapping A^{-1} possesses a certain type of continuity on $N = AM$. To this end, we require several new concepts and their properties.

Definition 1. The semi-deviation of a set m from a set m_0 is defined as the value

$$\beta(m, m_0) = \sup_{x \in m} \rho(x, m_0).$$

The geometric meaning of this value is quite simple. If we denote the ϵ-neighborhood of the set m_0 (i.e., the set $\{x : \rho(x, m_0) < \epsilon\}$) by $U(m_0, \epsilon)$, which coincides with the union of the ϵ-neighborhoods of all points in m_0, then the relation $\beta(m, m_0) < \epsilon$ is equivalent to $m \subset U(m_0, \epsilon)$. Obviously, if m consists of a single point x, then $\beta(x, m_0) = \rho(x, m_0)$.

Definition 2. A (multi-valued) mapping $F : Y \to X$ from one metric space Y to another is called β-continuous at the point y_0 if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for every y satisfying $\rho(y, y_0) < \delta$, the following inequality holds:

$$\beta(Fy, Fy_0) < \epsilon.$$

(Here Fy and Fy_0 are the full images of the points). A mapping is called β-continuous if it is β-continuous at every point of the space Y.

Geometrically, β-continuous means that for all points sufficiently close to a given point, their images lie within an ϵ-neighborhood of the full image of the given point. The following two properties follow directly from the definitions: A) A single-valued β-continuous mapping is continuous in the conventional sense; B) The superposition (composition) of two β-continuous mappings is also β-continuous, provided that in the first mapping, the set of images of each point is bicompact.

Theorem 1. If the operator A is closed on the set M (as defined above), then the image $N = AM$ is closed in Y, and the inverse mapping $A^{-1} : N \to M$ of the set N onto M is β-continuous. Note that the theorem refers to the full pre-images of points in the set M, not in $D(A)$.

Proof. We show the closedness of N similarly to [?]. Let $y \in \bar{N}$. Then there exists a sequence $y_n \in N$ such that $y_n \to y$ as $n \to \infty$. Take any of the pre-images of the point y_n, i.e., an element $x_n \in M$ (the existence of such x_n since $y_n \in N$ is certain). The sequence $\{x_n\}$ belongs to the compact set $\mathfrak{M}$, so there exists a convergent subsequence $x_{n_k} \to x \in \mathfrak{M}$. Their images also converge: $Ax_{n_k} = y_{n_k} \to y$. Since all $x_{n_k} \in D(A)$, by Definition 1, $x \in D(A)$ and $Ax = y$. From this, $x \in M$ and $y \in AM = N$; therefore, N is closed. V. K. Ivanov brought the last condition, which was absent in [?], to my attention.

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In passing, we have effectively proven that for any convergent sequence $y_n \in N$ ($y_n \to y_0$ as $n \to \infty$) and regardless of how the pre-images $x_n \in A^{-1}y_n$ are chosen, the set of limit points of the sequence $\{x_n\}$ is non-empty and belongs to the full pre-image $A^{-1}y_0$.

The β-continuity of the mapping A^{-1} at the point y_0 follows from this. Indeed, if this were not the case, then for some $\epsilon > 0$ there would exist $y_i \in N$ ($i = 1, 2, ...$) such that although $y_i \to y_0$ as $i \to \infty$, the sets $A^{-1}y_i$ do not lie entirely within the neighborhood $U(A^{-1}y_0, \epsilon)$. That is, points $x_i \in A^{-1}y_i$ could be found outside this neighborhood. The sequence $\{x_i\}$ must have limit points; otherwise, it would contradict the established property. However, it cannot have limit points either, because they would have to lie outside $U(A^{-1}y_0, \epsilon)$ (since the complement $X \setminus U(A^{-1}y_0, \epsilon)$ is closed), which is impossible. Since the element y_0 is arbitrary, the theorem is fully proven.

Corollary. In the case of a single-valued mapping $A^{-1} : N \to M$, it is continuous (by virtue of property A).

Thus, in the case of metric spaces, Theorem 1 generalizes the topological theorem from [?]. Even if A is continuous and $\mathfrak{M}$ is compact, the stronger α-continuity cannot be proven. The significance of the proven theorem is that if, instead of the exact value y, a sufficiently close right-hand side y_δ of (1) is known, then every pre-image $x \in A^{-1}y_\delta$ approximates some exact solution of equation (1) with a given level of error.

The requirement that the initial data belong to the set N is quite burdensome and, as a rule, cannot be verified effectively (usually it is assumed a priori if the compact set $\mathfrak{M}$ is chosen appropriately). The following concepts help to eliminate this requirement.

Definition 3. A (metric) projection of a point y onto a closed set N of a metric space is any point $y' \in N$ for which

$$\rho(y, y') = \rho(y, N).$$

We shall call the set $Py = \{y' : y' \in N, \rho(y, y') = \rho(y, N)\}$ the full projection of y onto N. In this section, we assume everywhere that $Py \neq \emptyset$. If we denote by $S[y, \rho]$ the sphere $\{y' : \rho(y, y') = \rho\}$, then clearly $Py = N \cap S[y, \rho]$. From this, it is evident that Py is closed. By assumption, it is non-empty. We thus have a (generally multi-valued) projection $P : Y \to N$ of the entire space onto its closed subset N.

5. Element

An element $x' \in M$ is called a quasi-solution of equation (1) on the set M (or on a compact set M) if:

$$\rho(Ax', y) = \min_{x \in M} \rho(Ax, y).$$

In other words, a quasi-solution is the preimage of some projection of the point y onto the set AM. The (complete) set of quasi-solutions is evidently the full preimage $A^{-1}\mathcal{P}y = A^{-1}\Phi y = Ty$ of the set of projections. It should be noted that Theorem 2 in [3] is erroneous, as demonstrated by counterexamples. Theorem 1 of the present section confirms the validity of all other results in [3], with the exception of the final conclusion of Theorem 5.

From this, it follows, among other things, that the set of quasi-solutions is non-empty and coincides with the set of exact solutions of equation (1) if such solutions exist—that is, if $y \in AM$.

Theorem. A non-empty metric projection P of a metric space U onto a subset M is β-continuous at every point $y \in U$.

Proof. Let $y \in U$ and $\epsilon > 0$. Since Py is non-empty, there exists a $\delta < \epsilon/2$. According to (3), $Py = N \cap S(y, \delta) \subset S(y, \epsilon) \subset U(Py, \epsilon)$, which was to be proved. For this theorem to hold, it is evidently sufficient that there exists a neighborhood of the element y, every point of which has a (non-empty) projection in M. If M is a compact set, the mapping is β-continuous on the entire space (see [3]).

Definition.

If for any

If its metric projection exists and is unique, the set is called a Chebyshev set in the space.

Corollary. The projection of a space onto a Chebyshev set is continuous at the points of that set.

From the representation given in (4), Theorems 1 and 2, and property B) of β-continuous mappings, we derive the fundamental theorem of this section:

Theorem. The mapping T, which assigns to each $y \in Y$ the complete set of quasi-solutions m on a compact set M for equation (1) with a closed operator A, is β-continuous at every element $y \in Y$. The projections of each point y onto N form a non-empty compact set (specifically, if N is a Chebyshev set).

The condition $y \in U(y_\delta, \delta)$ is no longer essential here. However, if the minimum residual exceeds the error in the initial data—that is, if $\min \rho(Ax, y) > \rho(y, y_\delta)$ —then using a quasi-solution instead of the exact solution generally makes no sense. In such a case, one must either select a different element y (which is often difficult in practice) or choose a different compact set M (specifically, a broader one) instead of M_0.

Geometrically, the last theorem implies that for $y \in U(y_\delta, \delta)$, the set of quasi-solutions $m = Ty$ is such that for each $x \in m$, there corresponds (at least one) $x_\delta \in M_0$ such that $\rho(Ax, Ax_\delta) \leq 2\delta$. However, it is not necessarily true that for every $x_\delta \in M_0$ there must exist an $x \in m$; that is, not every exact solution (if multiple exist in M) is guaranteed to have an ϵ-approximation among the quasi-solutions. Conversely, if the exact solution x_δ is unique, then any element $x \in m$ represents its ϵ-approximation.

Corollary. If the mapping $T : Y \to M$ in the theorem is single-valued—that is, if $P : Y \to N$ and $A^{-1} : N \to M$ are uniquely defined operations—then the quasi-solution on the compact set M depends continuously on the right-hand side of equation (1) at any point $y \in Y$.

We emphasize once more that···

results

The statements in this section are conditional: they are derived under the assumption that there exists a non-empty compact set of metric projections onto the set A. However, for a set closed in a general metric space, this condition is not always satisfied. To satisfy this assumption, we will introduce a number of additional properties below. Furthermore, we will not subsequently require that $y_0 \in F$, i.e., that an exact solution exists within the compact set $\mathfrak{M}$.

Properties of Metric Projection and the Theory of Quasi-solutions

In this section, we determine the conditions under which the existence of a projection onto F implies its uniqueness.

Definition 7. A linear metric space Y (generally with a metric that is not translation-invariant) is called quasi-rotund if no sphere $S(y_0, \rho)$ contains any straight-line segment $[y_1, y_2] = \{y : y = \alpha y_1 + (1-\alpha) y_2, 0 \leq \alpha \leq 1\}$. This requirement is formally weaker than the rotundity (strict convexity) of the space, but in many cases, it coincides with it.

Lemma 1. For the complete projection of every point in a linear metric space Y onto any convex set $F \subset Y$ to contain no more than one element, it is necessary and sufficient that Y be quasi-rotund.

Proof. The necessity of the condition is obvious, as otherwise, one could take the segment and the point y_0 from Definition 7. Conversely, suppose there exist $y_1, y_2 \in F$ such that $y_1 \neq y_2$ and $y_1, y_2 \in P_F(y_0)$. Denoting

$$\rho(y_0, F) = \rho, \quad V(y_0, \rho) = \{y : \rho(y, y_0) \leq \rho\},$$

we have, analogously to (3): $P_F(y_0) = F \cap V(y_0, \rho)$. As the intersection of convex sets, $P_F(y_0)$ is convex and therefore contains the entire segment $[y_1, y_2]$. By the definition of a projection, $\rho(y, y_0) = \rho$ for any $y \in [y_1, y_2]$, i.e., $[y_1, y_2] \subset S(y_0, \rho)$. This violates Definition 7, and the lemma is proved.

A special case of linear metric spaces is (linear) normed spaces. For our purposes, they are of interest because in such spaces, quasi-rotundity coincides with the well-known property of rotundity (strict convexity).

Definition 8.

8. Normed Space

A normed space is called strictly convex (or rounded) if no open segment of an arbitrary ball $V(y_0, \rho)$ intersects its boundary $S(y_0, \rho)$. In a Banach space, it is sufficient to require this property for a single ball, such as the unit ball (see [?], p. 187).

Lemma. The strict convexity of a normed space is equivalent to its quasi-strict convexity.

Proof. It is quite obvious that quasi-strict convexity follows from strict convexity. We prove the converse implication by assuming that Definition 8 does not hold for some ball $V(y_0, \rho)$. That is, there exist distinct $y_1, y_2 \in V(y_0, \rho)$ such that $y_\alpha = \alpha y_1 + (1-\alpha) y_2 \in S(y_0, \rho)$ for some $0 < \alpha < 1$. Then:

$$\|y_\alpha - y_0\| \leq \alpha \|y_1 - y_0\| + (1-\alpha)\|y_2 - y_0\| \leq \alpha\rho + (1-\alpha)\rho = \rho.$$

This implies that the internal inequalities must, in fact, be equalities; i.e., $y_1, y_2 \in S(y_0, \rho)$. Thus, the endpoints of any segment in $V(y_0, \rho)$ containing an interior point on $S(y_0, \rho)$ must themselves lie on $S(y_0, \rho)$. Consequently, the entire segment lies on the boundary, and Definition 7 is violated, which was to be proved.

Since the projection of a point onto a compact set always exists, the proven lemmas imply that every convex compact set is a Chebyshev set in any (quasi-)strictly convex linear space. We now turn to the question of the existence of projections onto a closed convex set.

Lemma (Liskovets). In a reflexive space Y, the metric projection of every point onto an arbitrary closed convex set exists.

Proof. Let $d(y, M) = \rho$ and consider the set

$$M_1 = M \cap V(y, \rho + 1).$$

Since the space, being a Banach space, is locally convex, and the set M_1 is convex and closed, it is also weakly closed ([?], p. 457). In view of its boundedness in a reflexive Banach space, it is weakly compact ([?], p. 461). According to Shmulyan' s theorem ([?], p. 469), it follows that any decreasing sequence of non-empty closed convex subsets has a non-empty intersection.

M k = N n V (y , p + V k) (* = 1, 2 , , . .)

must have a non-empty intersection, which obviously represents the complete projection of the point.

Remark 1. It is easy to see that Lemma 3 remains valid in the case where M is the union of a finite number of convex sets (we shall call such a set "finitely-convex") and is closed. By Theorem 2, the projection is β-continuous at these points. We now apply these results to equation (1). To this end, we shall henceforth assume that the operator A is linear, i.e., additive and homogeneous on the domain $D(A)$. Such an operator maps a convex set into a convex set (see, for example, [?], p. 444); therefore, Theorem 3 takes the following form.

Theorem 4. Let X be a metric space and Y be a reflexive Banach space. Let A be a linear closed operator from X into Y such that the set M is finitely-convex. Then the set of quasi-solutions on M depends β-continuously on the right-hand side of (1) at every point y, provided that all P_y are compact. A sufficient condition for M to be finitely-convex is that the compact set $\mathfrak{M}$ possesses this property.

8. By comparing the lemmas from the preceding sections, we can confirm that in a strictly convex (round) reflexive Banach space, every closed convex set is a Chebyshev set (this is characteristic of this class of spaces, see [?]). In the general case, Theorem 4 also holds here, but this time in an unconditional form, since P_y consists of a finite number of points. In the case of a convex and single-valued mapping A^{-1}, the theorem can be further refined.

Corollary. If, under the conditions of Theorem 4, the space Y is also strictly convex, the set M is convex, and the inverse operator A^{-1} exists, then the quasi-solution of equation (1) on M depends continuously on its right-hand side at every point y.

Remark.

2. In the case of $\widehat{\ }$-space

As noted in [?] (p. 64), one can achieve uniqueness of the mapping by considering equation (1) on the quotient space X/X_0, where X_0 is the linear subspace of solutions to the homogeneous equation $Ax = 0$. This subspace is closed due to the closedness of the operator A. Under these conditions, the operator remains linear and closed. However, the disadvantage of this approach lies in the necessity of considering all (quasi-)solutions from the equivalence class simultaneously. Consequently, it becomes impossible to isolate a specific desired subset of solutions by enclosing them within a suitable compact set M.

9. Previous works

Results can be improved by imposing stricter requirements on the space Y. Thus far, even under the conditions of Corollary 4, where M is a closed convex Chebyshev set, we could only assert the continuity of the metric projection onto M at points belonging to the set M itself. Whether the projection is continuous throughout an entire rounded reflexive Banach space Y for a closed convex set M remains unknown in the general case.

Therefore, we consider a narrower class of spaces Y, the so-called E-spaces [?]. These are Banach spaces in which, for every closed convex set M, any sequence

$\{y_n\} \subset M$ converges if it satisfies the condition:

$$\lim_{n\to\infty} \|y_n\| = \inf_{y\in M} \|y\|$$

(A series of equivalent definitions is provided in [?] and [?]; uniformly convex spaces, and specifically complete Hilbert spaces, are E-spaces, which in turn are rounded and reflexive (see [?])). As shown in [?], the projection onto a closed convex set is continuous throughout the entire E-space and is, therefore, well-posed in the sense of Hadamard. From this, it is easy to deduce that in an E-space, the metric projection onto a closed finitely-convex set possesses β-continuity. Consequently, the following theorem holds:

Theorem 2. If, under the conditions of Theorem 1, Y is an E-space, then the set of quasi-solutions of equation (1) on M depends β-continuously on $y \in Y$.

Thus, we have eliminated the requirement $y \in T(M)$, so there is no longer a need to insist that a solution to equation (1) with the "exact" right-hand side exists within the chosen compact set.

Corollary 5. If, under the conditions of Corollary 4, Y is an E-space, then the quasi-solution of equation (1) on M depends continuously on $y \in Y$, and therefore the problem of finding such a quasi-solution is well-posed in the sense of Hadamard. This means that for a metric space X, this theorem generalizes the main result of [?]. Remark 2 also applies to this case.

Just as in [?], it is not difficult—given a convex M—to provide an estimate for the β-stability of quasi-solutions via the modulus of β-continuity of the mapping T at point y_0, which is understood as:

$$\omega(y_0, \delta) = \sup_{\|y-y_0\|\leq\delta} \beta(T_y, T_{y_0})$$

This specific value was chosen rather than the modulus of β-continuity over the entire Y because, by Theorem 1, it is infinitesimal as $\delta \to 0$. It is obvious that $\rho(T_y, T_{y_0}) \leq \rho(A^{-1}Py, A^{-1}Py_0) \leq \omega_A(\|Py - Py_0\|)$, and it is only necessary to estimate $\|Py - Py_0\|$ in terms of $\|y - y_0\|$; that is, to estimate the modulus of continuity of the projection operation P. For Hilbert spaces, such an estimate is given in [?]. It turns out that $\|Py - Py_0\| \leq \|y - y_0\|$ for $y, y_0 \in Y$, and therefore:

$$\rho(T_y, T_{y_0}) \leq \omega_A(\|Py_0, \|y - y_0\|), \quad y, y_0 \in Y.$$

3. APPROXIMATE CONSTRUCTION OF QUASI-SOLUTIONS

Let us now turn to the practical question of methods for finding quasi-solutions. In doing so, we shall restrict our consideration to the most important case, that of Hilbert space.

A. LISKOVETS. Let the set Φ be convex, and suppose there exists some increasing sequence of convex subsets $\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_n \subset \cdots \subset \Phi$. Then their images under a linear closed operator A also constitute an increasing sequence of convex closed sets: $A\Phi_1 \subset A\Phi_2 \subset \cdots \subset A\Phi_n \subset \cdots \subset A\Phi$. Let us denote

M o0 =UM i , N o0 = uN i =AM oa .

For the following discussion, it is crucial that N_0 is dense in N, i.e., $\bar{N}_0 = N$. In practice, however, direct verification of this condition can be quite difficult. Therefore, it is preferable to provide a corresponding sufficient condition expressed in terms of the space X.

Lemma 4

Let X be a metric space, Y be a Banach space, $A : X \to Y$ be a linear closed operator, and $M \subset X$ be a relatively compact, finitely convex set. If for some $M' \subset M$ and any segment $l \subset M$, the intersection $l \cap M'$ is dense in l, then the image $N' = A(M')$ is dense in $N = A(M)$.

Proof. It is clearly sufficient to consider convex sets M. Let us show that N possesses the same properties as M. Take arbitrary points $y_1, y_2 \in N$ and any of their pre-images $x_1, x_2 \in M$. We restrict the operator A to the segment $l = [x_1, x_2]$. This restriction is obviously a closed, one-to-one linear operator. According to Corollary 1, it has a continuous inverse on its image $A(l)$. Since the set $A(l)$ is one-dimensional, bounded, and closed in the metric of the Banach space Y, it is compact. Consequently, the operator A itself is continuous on l. Combined with condition (5), this implies that $A(l \cap M')$ is dense in $A(l)$. The relations $A(l \cap M') \subset N' \cap A(l)$ and $y_1, y_2 \in A(l) \subset N$ complete the proof.

We shall now assume that Y is a Hilbert space. For an arbitrary $y \in Y$, let y_n denote its full metric projections onto the previously constructed sets N_n.

Theorem

If the space Y is Hilbert, the sets N_n are convex, and N_0 is dense in N (specifically, if condition (5) is satisfied for M_0), then the projections of the point $y \in Y$ converge:

$$y_n \to y_0, \quad n \to \infty$$

Proof. By virtue of the results···

§ 2 Projections

exist and each consist of a single point. Let us denote

$P(y, N) = p(y, n) = p$, $P(y, M) = p(y, m) = p_1$.

It follows from the inclusions that $l + 1$. Therefore, the sequence $(i = 1, 2, ...)$ converges, and since it is dense in, it converges to p. The assertion of the theorem follows directly from this.

Indeed, by analogy with (3), $n \in N \cap L \cap S(p, J) \cap S(y, P_t)$. If $p = 0$ and $n = \{y\}$, then for $\{y, p\} \cup \{n\}$, if $p > 0$ and $\{y'\}$, where $y' \neq y$, we separate the sets (y, p) and N by a hyperplane h (this is possible by Eidelheit' s theorem [**?**], p. 42). This hyperplane is a supporting hyperplane to the point (y, p); that is, it is orthogonal to the segment $[y, y']$, since there can be no other supporting hyperplanes at point y due to the smoothness of the Hilbert space. Of the two resulting closed half-spaces, one contains the set, which we denote by H. Then $[S(y, H) \cap S(y, P)]$. We shall now show that for $p_1 - p_2$

H[§(y, Pi)CU(y' ,e) = U(n ,E).

To achieve this, we pass an arbitrary two-dimensional plane through $[y, y']$. This plane is clearly Euclidean. A chord of the circle $L \cap S(y, \rho)$ passes through the point y'. The point y' serves as the midpoint of this chord, since $y' \perp y$. Among the points in the truncated part of the circle, the endpoints of the chord are the most distant; the distance to them is equal to ρ, which completes the proof. Along the way, we obtain the convergence rate of the projections:

$\rho(y', \pi) < \sqrt{\rho^2 - \rho {o'}^2}$

If the sets in the theorem are assumed to be finitely convex, it is evident from the proof that they are β-convergent, i.e., $\rho(\pi_i, \pi) \to 0$ as $i \to \infty$.

Corollary 6. Under the conditions of the theorem, if the sets are finitely convex, each closed convex component of the set π is individually approximated, in the sense of this theorem, by the corresponding components of the sets π_i. A similar result was proven in [**?**] for a continuous operator A, which allowed for a significant reduction in many other requirements. Denoting the quasi-solutions of equation (1) on the sets M_i and M as x_i and x respectively, we arrive at the following conclusion, which is the primary result of this section.

Theorem. Under the conditions of the theorem and Corollary 6, given a linear operator A, the sets x_i are β-convergent to x. In particular, under the conditions of the corollary to the theorem, the quasi-solution x_i converges to the quasi-solution x as $i \to \infty$. This theorem is especially useful when the sets M_i are finite-dimensional. In such cases, the functional $\|Ax - f\|$ depends on a finite number of variables. There are many effective methods available for minimizing such functionals, which makes it possible to find the set of quasi-solutions x_i. Moreover, if the quasi-solution is unique, it is sufficient to find any single element x_i, as any such element will approximate the desired quasi-solution x. Comparing these results···

results

Based on the preceding sections, we can verify that:

$$P(\eta > \eta_0) < P\left(f(y_0) + \langle y - y_0, \nabla f(y_0)\rangle + \frac{1}{2}\rho\|y - y_0\|^2 > \eta_0\right)$$

as $t \to \infty$ for $y > y_0$ and $y \in Y$.

convex and finitely-convex

In conclusion, we note that the assumption regarding the inclusions $j > 1$ is not strictly required by the nature of the problem; it is used primarily to simplify the proof. Through a somewhat more complex derivation, it can be shown that the presented results remain valid even in the case where the lower limit of the sequence of sets $M_j \subset X$ (denoted again by M_∞) is considered, i.e., the set:

$$M_\infty = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} M_j$$

satisfies the conditions stated previously. In this case, it follows that the convergence in (6) is no longer required to be monotonic.

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Note: Figure translations are in progress. See original paper for figures.