

# ASYMPTOTICS OF THE NUMBER OF SOLUTIONS OF CERTAIN DIOPHANTINE INEQUALITIES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ASYMPTOTICS OF THE NUMBER OF SOLUTIONS OF CERTAIN DIOPHANTINE INEQUALITIES

*(Presented by Academician I. M. Vinogradov on 30 V 1966)*

Let  $\alpha, \beta$  be real numbers; let  $\chi(q)$  be a positive monotonically decreasing function defined for natural numbers  $q$ ; let  $\|x\| = \min(\{x\}, 1 - \{x\})$  be the distance from  $x$  to the nearest integer. It is known that for almost all  $\alpha$  the number of solutions of the inequality

$$\|\alpha q + \beta\| < \chi(q) \quad (1)$$

in integers  $q$ ,  $1 \leq q \leq Q$ , is asymptotically equal to  $2\Psi(Q) + O(\Psi^{1/2+\varepsilon}(Q))$ , where  $\Psi(Q) = \sum_{q=1}^Q \chi(q)$ , if  $\Psi(Q) \rightarrow \infty$  (see, for example, <sup>(5, 8, 11, 12)</sup>). It is of interest to give "individual" variants of this fact, i.e. to indicate concrete numbers  $\alpha$  (or classes of numbers  $\alpha$ ) for which the analogous assertion is true. Recently S. Lang <sup>(6, 7)</sup> proved for quadratic irrationalities  $\alpha$  and functions  $\chi(q)$  decreasing as  $q^{-1}$  or considerably more slowly than  $q^{-1}$ , that the number of solutions of (1) with  $\beta = 0$  has the asymptotic  $2\Psi(Q)$ . Lang's arguments are based on the arithmetic of quadratic fields and do not admit any generalization to the case of other  $\alpha$ .

In this note we show, in particular, that if  $\alpha$  satisfies the following condition (A): the number of solutions of the inequality  $\|\alpha q\| < q^{-1-\varepsilon}$  is finite for every  $\varepsilon > 0$ , then for any  $\beta$  and any  $\sigma$ ,  $0 < \sigma < 1$ , the number of solutions of the inequality

$$\|\alpha q + \beta\| < q^{-\sigma}, \quad 1 \leq q \leq Q, \quad (2)$$

is asymptotically equal to

$$\frac{2}{1-\sigma} Q^{1-\sigma} + O(Q^{(1-\sigma)/2+\varepsilon}).$$

The following more general result is true.

**Theorem.** Let  $\chi(t)$  be a positive continuously differentiable function defined for  $t \geq 1$ , whose derivative satisfies the conditions: 1)  $\chi'(t) < 0$  for all  $t \geq 1$ ,  $|\chi'(t)|$  tends monotonically to zero as  $t \rightarrow \infty$ ; 2)  $|\chi'(t)| \gg t^{-2}$  ( $t \gg 1$ ), and if  $t < t_1 \leq t$ , then  $|\chi'(t)| \ll |\chi'(t_1)|$ . Then, for  $\alpha$  satisfying condition (A), and for any  $\beta$ , the number of solutions of inequality (1) is asymptotically equal to

$$2\Psi(Q) + O(\Psi(Q)\chi^{-1}(Q)|\chi'(Q)|^{1/2}Q^\varepsilon) + O(\chi(Q)|\chi'(Q)|^{-1/2}),$$

where

$$\Psi(Q) = \int_1^Q \chi(t) dt.$$

This theorem is an immediate consequence of the well-known fact of the strong uniform distribution of the fractional parts  $\|\alpha q + \beta\|$  for  $\alpha$  satisfying condition (A) (see Lemma 1), and of Lemma 2 below, which makes it possible to determine the asymptotics of the number of solutions of the inequality  $a_q < \chi(q)$ , if it is known that the sequence  $a_q$  is well distributed on some interval  $(0, \lambda)$ ,  $\lambda > 0$ .

It is known that condition (A) is satisfied by irrational algebraic numbers <sup>(10, 2)</sup>, and also by numbers of the form  $e^r$  ( $r \neq 0$ ) and  $\ln r$  ( $r \neq 0, 1$ ), where  $r$  is a rational number <sup>(3, 4, 9)</sup>.

**Lemma 1.** Let  $\alpha$  be a real number satisfying condition (A); let  $\beta$  be any real number; let  $I$  be an arbitrary interval inside the interval  $(0, \frac{1}{2})$ ; and let  $N_Q(I)$  be the number of fractional parts  $\|\alpha q + \beta\|$ ,  $1 \leq q \leq Q$ , lying in  $I$ . Then

$$N_Q(I) = 2|I|Q + O(Q^\varepsilon),$$

where  $|I|$  is the length of the interval  $I$ , and the estimate is uniform over all intervals  $I$ .

The proof of this lemma is obtained without difficulty by applying I. M. Vinogradov's "little cups" method <sup>(1)</sup>, p. 260).

**Lemma 2.** Let  $a_q$  ( $q = 1, 2, \dots$ ) be a sequence of real numbers lying in the interval  $(0, \lambda)$ , where  $\lambda > 0$  is some fixed number; let  $I$  be an arbitrary interval inside the interval  $(0, \lambda)$ ; let  $N_Q(I)$  be the number of those  $a_q$ ,  $1 \leq q \leq Q$ , which lie in  $I$ , and suppose that

$$N_Q(I) = \lambda^{-1}|I|Q + R_Q(I), \quad |R_Q(I)| \leq R(Q), \quad (3)$$

where the quantity  $R(Q)$  does not depend on  $I$ , and  $R(Q)$  increases monotonically to infinity. Finally, let  $\chi(t)$  be a function satisfying the conditions of the theorem. Then the number  $N(Q)$  of those  $a_q$  which satisfy the inequality  $a_q < \chi(q)$ ,  $1 \leq q \leq Q$ , is equal to

$$\lambda^{-1}\Psi(Q) + L(Q),$$

$$L(Q) = O(\Psi(Q)\chi^{-1}(Q)|\chi'(Q)|^{1/2}R(Q) + \chi(Q)|\chi'(Q)|^{-1/2}).$$

**Proof.** Define the numbers  $p_k$  from the conditions  $p_1 = 1$ ,

$$p_{k+1} - p_k = |\chi'(p_k)|^{-1/2}.$$

It is clear that the numbers  $p_k$  are positive and the difference  $p_{k+1} - p_k$  increases monotonically. Denote by  $N_0(k)$  the number of those  $a_q$  for which the inequality  $a_q < \chi(q)$  holds under the condition  $p_k \leq q < p_{k+1}$ ; by  $N_1(k)$ , the number of those  $a_q$  for which  $a_q < \chi(p_k)$  under  $p_k \leq q \leq p_{k+1}$ ; and, analogously, denote by  $N_2(k)$  the number of those  $a_q$  for which  $a_q < \chi(p_{k+1})$  under  $p_k \leq q < p_{k+1}$ . Since the function  $\chi(t)$  decreases monotonically, we have  $N_2(k) \leq N_0(k) \leq N_1(k)$ , i.e.

$$N_0(k) = N_1(k) - \theta(N_1(k) - N_2(k)), \quad 0 \leq \theta \leq 1.$$

Since the difference  $N_1(k) - N_2(k)$  is the number of those  $a_q$  which fall into the interval  $(\chi(p_{k+1}), \chi(p_k))$ , by the condition of the lemma (3) this difference does not exceed the quantity

$$O((\chi(p_k) - \chi(p_{k+1}))(p_{k+1} - p_k) + R(p_{k+1})).$$

Define  $k_0$  from the condition  $p_{k_0} \leq Q < p_{k_0+1}$ . Then, since

$$N_1(k) = \lambda^{-1}\chi(p_k)(p_{k+1} - p_k) + O(R(p_{k+1})),$$

we have

$$\begin{aligned} N(Q) &= \sum_{k=1}^{k_0-1} N_1(k) + L_1(Q) = \\ &= \sum_{k=1}^{k_0-1} \lambda^{-1}\chi(p_k)(p_{k+1} - p_k) + L_2(Q) = \lambda^{-1}\Psi(Q) + L_3(Q), \end{aligned}$$

where all quantities  $L_j(Q)$  have order

$$O\left(\sum_{k=1}^{k_0-1} (\chi(p_k) - \chi(p_{k+1}))(p_{k+1} - p_k) + k_0 R(Q) + \chi(p_{k_0})(Q - p_{k_0})\right). \quad (4)$$

Let  $x, y$  be real numbers satisfying  $p_k \leq x < y \leq p_{k+1}$ . By the choice of the numbers  $p_k$  and the monotone variation of the functions  $\chi(t)$ ,  $|\chi'(t)|$ , we find

$$\begin{aligned} 0 &< (\chi(x) - \chi(y))(y - x) \leq |\chi'(x)|(y - x)^2 \leq \\ &\leq |\chi'(p_k)|(p_{k+1} - p_k)(y - x) = (y - x)(p_{k+1} - p_k)^{-1} \leq 1. \end{aligned}$$

Therefore

$$\chi(p_{k_0})(Q - p_{k_0}) \leq \chi(Q)(Q - p_{k_0}) + 1 \leq \chi(Q)|\chi'(Q)|^{-1/2} + 1$$

and the remainder term (4) will be a quantity

$$O(k_0 R(Q) + \chi(Q) |\chi'(Q)|^{-1/2}).$$

Let us estimate the quantity  $k_0$ . Setting  $k_1 = [k_0/2]$ , we have

$$\begin{aligned} \int_{q_{k_1}}^{q_{k_0}} \chi(t) dt &\geq \chi(Q)(p_{k_0} - p_{k_1}) = \chi(Q) \sum_{k_1 \leq k < k_0} (p_{k+1} - p_k) = \\ &= \chi(Q) \sum_{k_1 \leq k < k_0} |\chi'(p_k)|^{-1/2} \geq \chi(Q) |\chi'(p_{k_1})|^{-1/2} (k_0 - k_1). \end{aligned}$$

By virtue of the conditions imposed on  $|\chi'(t)|$ , we find

$$\frac{p_{k+1}}{p_k} = 1 + p_k^{-1} |\chi'(p_k)|^{-1/2} \ll 1, \quad p_{k+1} \ll p_k;$$

therefore  $Q \ll p_{k_1}$ , and then  $|\chi'(p_{k_1})| \ll |\chi'(Q)|$ . Thus,

$$k_0 \ll \chi^{-1}(Q) |\chi'(Q)|^{1/2} \int_1^Q \chi(t) dt,$$

which completes the proof of Lemma 2.

In the case of simultaneous approximations

$$\max(\|a_1 q + \beta_1\|, \dots, \|a_n q + \beta_n\|) < \chi(q) \tag{5}$$

analogous results hold, as is not difficult to establish by only slightly developing the arguments. Condition (A) here must be replaced by the condition: the number of solutions of the inequality

$$\|a_1 q\| \cdot \|a_2 q\| \dots \|a_n q\| < q^{-1-\varepsilon}$$

in integers  $q > 0$ , for every  $\varepsilon > 0$ , is finite; or, equivalently, the number of solutions of the inequality

$$\|a_1 a_1 + \alpha_2 a_2 + \dots + a_n a_n\| > (a'_1 a'_2 \dots a'_n)^{-n-\varepsilon}$$

in integers  $a_i$  ( $i = 1, 2, \dots, n$ ), where  $a'_i = \max(1, |a_i|)$ , is finite for every  $\varepsilon > 0$ . For example, one may take  $\alpha_i = e^{r_i}$ , where  $r_i \neq 0$  are pairwise distinct rational numbers <sup>(4)</sup>. Under these conditions, the number of solutions of the system of

inequalities (5) with  $\chi(q) = q^{-\sigma}$ ,  $0 < \sigma < 1/n$ ,  $1 \leq q \leq Q$ , is asymptotically equal to

$$\frac{2^n}{1 - n\sigma} Q^{1-n\sigma} + O\left(Q^{\frac{1-n\sigma}{2} + \varepsilon}\right).$$

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## REFERENCES

- <sup>1</sup> I. M. Vinogradov, *Selected Works*, Moscow, 1952.
- <sup>2</sup> J. Cassels, *An Introduction to Diophantine Approximation*, Moscow, 1961.
- <sup>3</sup> A. Baker, *Acta Arithm.*, **10**, 315 (1964).
- <sup>4</sup> A. Baker, *Canad. J. Math.*, **17**, No. 4, 616 (1965).
- <sup>5</sup> P. Erdős, *Acta Arithm.*, **5**, 359 (1959).
- <sup>6</sup> S. Lang, *Am. J. Math.*, **87**, No. 2, 481 (1965).
- <sup>7</sup> S. Lang, *Am. J. Math.*, **87**, No. 2, 488 (1965).
- <sup>8</sup> W. J. LeVeque, *Trans. Am. Math. Soc.*, **94**, 130 (1959).
- <sup>9</sup> K. Mahler, *J. reine u. angew. Math.*, **166**, 137 (1932).
- <sup>10</sup> K. F. Roth, *Mathematika*, **2**, 1 (1955).
- <sup>11</sup> W. M. Schmidt, *Canad. J. Math.*, **12**, 619 (1960).
- <sup>12</sup> W. M. Schmidt, *Trans. Am. Math. Soc.*, **110**, 493 (1964).

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