FUNDAMENTAL BOUNDARY-VALUE PROBLEMS OF POTENTIAL THEORY FOR A SPACE WITH AN ELLIPTIC SLIT. PRESSURE OF AN ELLIPTIC PUNCH ON AN ELASTIC HALF-SPACE

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Abstract

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FUNDAMENTAL BOUNDARY-VALUE PROBLEMS OF POTENTIAL THEORY FOR A SPACE WITH AN ELLIPTIC SLIT. PRESSURE OF AN ELLIPTIC PUNCH ON AN ELASTIC HALF-SPACE

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In 1896 A. Sommerfeld developed a method for solving a number of boundary-value problems of potential theory by means of a many-valued analogue of the fundamental solution of Laplace' s equation, branching around a straight line ([1]). The concepts introduced by him of a Riemann space and of the Green' s function of a Riemann space were used by the author ([2]) to solve the fundamental boundary-value problems of potential theory for a space with a "slit," understood as a cut along an unclosed surface of Lyapunov type bounded by a smooth curve. In the case of a plane slit it was shown that the solution of the Dirichlet and Neumann problems is expressed explicitly in terms of the Green' s function of a two-sheeted Riemann space whose branch line is the curve bounding the slit ([3]). Green' s functions were constructed by A. Sommerfeld ([1]) for a rectilinear line of branching, and by E. Hobson ([4]) and Z. Neishtedter ([5]) for a circular line of branching. In the present work we give the Green' s function, constructed by the author, of a two-sheeted Riemann space with an elliptic line of branching, and present its application to the explicit solution of the Dirichlet and Neumann problems for a space with an elliptic slit. The solutions obtained are used, according to the scheme of L. A. Galin ([6]), in solving the problem of the pressure of an elliptic, in plan, punch on an elastic half-space in the absence of friction forces.

1. Let us consider in the plane xoy an ellipse with center at the origin, whose major and minor semiaxes are respectively a and b. We introduce a coordinate system associated with this ellipse. To this end, through the point $M(x, y, z)$ and the z-axis we draw a plane and denote by A and B the points at which this plane intersects our ellipse (for definiteness we shall assume that A is the point nearer to M, and B the farther one). We take as the new coordinates of the point M the numbers $\rho = \ln(AM/BM)$,

$\theta = \angle AMB$, and φ, the polar angle of the point M. The Cartesian coordinates of the point M are expressed in terms of the just-introduced coordinates (ρ, θ, φ) by the formulas:

$$x = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} \cos \varphi \operatorname{sh} \rho / (\operatorname{ch} \rho - \cos \theta),$$

$$y = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} \sin \varphi \operatorname{sh} \rho / (\operatorname{ch} \rho - \cos \theta),$$

$$z = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} \sin \theta / (\operatorname{ch} \rho - \cos \theta).$$

It is obvious that the toroidal coordinate system is obtained from ours for $a = b$.

The Green' s function of a two-sheeted Riemann space with an elliptic line of branching has the form

$$v(M, N) = \frac{1}{R} \left[\frac{1}{2} + \frac{1}{\pi} \arcsin \frac{\cos(\theta - \theta_0)/2}{\operatorname{ch} \alpha / 2} \right],$$

where R is the distance between the points M and N (in what follows the Cartesian coordinates of the points M and N will be used along with those introduced above: $M(x, y, z) \equiv M(\rho, \theta, \varphi)$, $N(x_0, y_0, z_0) \equiv N(\rho_0, \theta_0, \varphi_0)$), and $\operatorname{ch} \alpha$ is determined by the relation

$$\begin{aligned} \operatorname{ch} \alpha = & \frac{1}{2} \sqrt{\frac{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0}} (\operatorname{ch} \rho \operatorname{ch} \rho_0 - \operatorname{ch} \rho \cos \theta_0 + \operatorname{ch} \rho_0 \cos \theta) \\ & + \frac{1}{2} \sqrt{\frac{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0}{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} (\operatorname{ch} \rho \operatorname{ch} \rho_0 - \operatorname{ch} \rho_0 \cos \theta + \operatorname{ch} \rho \cos \theta_0) \\ & - \frac{\left(\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} - \sqrt{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0} \right)^2}{2 \sqrt{\left(a^2 \cos^2 \varphi + b^2 \sin^2 \varphi \right) \left(a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0 \right)}} \cos \theta \cos \theta_0 \\ & - \cos(\varphi - \varphi_0) \operatorname{sh} \rho \operatorname{sh} \rho_0. \end{aligned}$$

2. Denote by S the region $\{x^2/a^2 + y^2/b^2 \leqslant 1,\ z = 0\}$, by Σ the region $\{x^2/a^2 + y^2/b^2 \geqslant 1,\ z = 0\}$, and by E^3 three-dimensional space.

Consider the following boundary-value problems:

The Dirichlet problem—to determine a function harmonic in $E^3 \setminus S$, regular at infinity, and taking the prescribed values f_+ and f_- when approaching S

from above and from below (i.e., in the direction of the positive and negative z-axis, respectively).

The Neumann problem–to determine a function harmonic in $E^3 \setminus S$, regular at infinity, whose normal derivative takes the prescribed values f_+ and f_- when approaching S from above and from below. Here f_+ and f_- are continuous functions defined on S and coinciding on the contour bounding S.

The solution of the Dirichlet problem formulated above has the form

$$\begin{aligned} u(M) = \frac{1}{2\pi} \iint_S \Bigg\{ \frac{z}{R_0^3} \left[\frac{1}{2} + \frac{1}{\pi} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} \right] \\ + \frac{1}{2\pi R_0} \frac{\sqrt{2}\,(\operatorname{ch} \rho_0 + 1) \cos \theta/2}{\sqrt{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0} \sqrt{\operatorname{ch} \alpha_1 + \cos \theta}} \Bigg\} f_+(N)\, d_{NS} \\ - \frac{1}{2\pi} \iint_S \Bigg\{ \frac{z}{R_0^3} \left[\frac{1}{2} - \frac{1}{\pi} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} \right] \\ - \frac{1}{2\pi R_0} \frac{\sqrt{2}\,(\operatorname{ch} \rho_0 + 1) \cos \theta/2}{\sqrt{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0} \sqrt{\operatorname{ch} \alpha_1 + \cos \theta}} \Bigg\} f_-(N)\, d_{NS}. \end{aligned}$$

For $f_+ = f_- = f$,

$$u(M) = \frac{1}{\pi^2} \iint_S \left\{ \frac{z}{R_0^3} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} + \frac{1}{2R_0} \frac{\sqrt{2}\,(\operatorname{ch} \rho_0 + 1) \cos \theta/2}{\sqrt{a^2 \cos^2 \varphi_0 + b^2 \sin^2 \varphi_0} \sqrt{\operatorname{ch} \alpha_1 + \cos \theta}} \right\} f(N)\, d_{NS}.$$

The solution of the Neumann problem has the form

$$u(M) = -\frac{1}{2\pi} \iint_S \frac{1}{R_0} \left[\frac{1}{2} + \frac{1}{\pi} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} \right] f_+(N)\, d_{NS} + \frac{1}{2\pi} \iint_S \frac{1}{R_0} \left[\frac{1}{2} - \frac{1}{\pi} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} \right] f_-(N)\, d_{NS}.$$

For $f_+ = f_- = f$,

$$u(M) = -\frac{1}{\pi^2} \iint_S \frac{1}{R_0} \arcsin \frac{\sin \theta/2}{\operatorname{ch} \alpha_1/2} f(N)\, d_{NS}.$$

Here R_0 is the distance between the points $M(x, y, z) \equiv M(\rho, \theta, \varphi)$ and $N(x_0, y_0, 0) \equiv N(\rho_0, \pi, \varphi_0)$; $\operatorname{ch} \alpha_1 = \operatorname{ch} \alpha\big|_{\theta_0 = \pi}$.

It is possible, in a completely analogous way, to formulate the Dirichlet and Neumann problems for an exterior elliptic slit and to obtain analogous formulas for their solution.

3. The formulas obtained above were used to solve a number of problems in the theory of elasticity (the stress state of a body weakened by an elliptic internal or external slit, the contact problem of the theory of elasticity for an elliptic punch, etc.). We shall restrict ourselves only to the formulation of several results relating to the problem of the pressure of an elliptic punch in plan on an elastic half-space in the absence of frictional forces. Let the equation of the surface of the punch have the form $z = f(x, y)$. Then the pressure acting on the contact area can be computed by the formula

$$p(M_0) = \frac{E}{2\pi^2(1-\nu^2)}\frac{\partial}{\partial z}\left\{\iint\limits_S \left[\frac{z}{R_0^3}\arcsin\frac{\sin\theta/2}{\operatorname{ch}\alpha_1/2} + \frac{\sqrt{2}(\operatorname{ch}\rho_0+1)\cos\theta/2}{2R_0\sqrt{a^2\cos^2\varphi_0+b^2\sin^2\varphi_0}\sqrt{\operatorname{ch}\alpha_1+\cos\theta}}\right] f(N)\, d_{NS}\right\}$$

If the surface bounding the punch has derivatives at the points located on the edges of the contact area, then the pressure under the punch is expressed by the formula

$$p(M_0) = -\frac{E}{2\pi^2(1-\nu^2)}\iint\limits_S \frac{1}{R_{00}}\operatorname{arc\,tg}\frac{1}{R_{00}}\times$$

$$\times\sqrt{\frac{(a^2\cos^2\varphi+b^2\sin^2\varphi-r^2)(a^2\cos^2\varphi_0+b^2\sin^2\varphi_0-r_0^2)}{\sqrt{(a^2\cos^2\varphi+b^2\sin^2\varphi)(a^2\cos^2\varphi_0+b^2\sin^2\varphi_0)}}}\,\Delta f(r_0,\varphi_0)\, r_0\, dr_0\, d\varphi_0,$$

where $ds = r_0 dr_0 d\varphi_0$,

$$R_{00} = \sqrt{(x-x_0)^2+(y-y_0)^2} = \sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)},$$

$$M_0 = M(r_0,\varphi_0,0) \equiv M(x,y,0).$$

A load situated outside the punch produces under the punch an additional pressure

$$p(M_0) = \frac{1}{\pi^2}\iint\limits_\Sigma \frac{1}{R_{00}^2}\sqrt{\frac{r_0^2-a^2\cos\varphi_0-b^2\sin^2\varphi_0}{a^2\cos^2\varphi+b^2\sin^2\varphi-r^2}}\times$$

$$\times\sqrt[4]{\frac{a^2\cos^2\varphi+b^2\sin^2\varphi}{a^2\cos^2\varphi_0+b^2\sin^2\varphi_0}}\, q(r_0,\varphi_0)\, ds.$$

The formulas presented have been obtained with the aid of the well-known method of L. A. Galin, developed by him for punches circular in plan [6]. The results of L. A. Galin are obtained when $a = b$.

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Note: Figure translations are in progress. See original paper for figures.