ESTIMATES OF THE GREEN MATRIX OF A HOMOGENEOUS PARABOLIC BOUNDARY-VALUE PROBLEM

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Abstract

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ESTIMATES OF THE GREEN MATRIX OF A HOMOGENEOUS PARABOLIC BOUNDARY-VALUE PROBLEM

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In recent years a detailed investigation has been carried out of boundary-value problems for linear parabolic systems in various function spaces ([1–5]). It has thereby been established that the algebraic Lopatinskii condition imposed on the connection of the boundary operators with the system of equations guarantees the correct solvability of these problems (in what follows we shall call such problems parabolic).

Here it is established that the operator solving a homogeneous parabolic boundary-value problem is an integral operator, and sharp estimates up to the boundary are obtained for the derivatives of the kernel of this operator—the Green matrix. These results are then applied to establishing analogous estimates of the Green matrix of elliptic boundary-value problems generated by parabolic ones.

For simplicity the exposition is given for systems of first order with respect to t; all results are valid for Petrovsky-parabolic systems of any order.

1. Let Q be the cylindrical domain $[0, T] \times \Omega$, where T is a finite positive number and Ω is a finite or infinite domain in the space E_n. The boundary S of the domain Ω may also be infinite and is assumed to belong to the class C^l with some sufficiently large $l > 0$ (for the definition of the class of surfaces C^l and of all other classes of functions occurring below, see ([4])). In the domain Q there is given a Petrovsky-parabolic system

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{|k|\le 2b} A_k(t,x) D_x^k u = f(t,x), \tag{1}$$

the initial condition

$$u\big|_{t=0} = \varphi(x) \tag{2}$$

and the boundary conditions

$$B_\nu(t,x,D_x)u\big|_\Gamma \equiv \sum_{j=1}^{N}\sum_{|k|\le r_\nu} b_{\nu j}^{(k)}(t,x)D_x^k u_j\big|_\Gamma = 0, \tag{3}$$

where $\Gamma = (0,T]\times S,\ r_\nu \le 2b-1,\ \nu = 1,2,\dots,bN$*.

The following main theorem holds.

Theorem 1. 1) If the coefficients $A_k(t,x)$ of the system (1) belong to the class $C_{x,t}^{\alpha,\alpha/2b}(Q)$, the coefficients $b_{\nu j}^{(k)}(t,x)$ of the boundary operators belong to the class $C_{x,t}^{2b-r_\nu+\alpha,\ \frac{1}{2b}(2b-r_\nu+\alpha)}(\Gamma)$, and $S\in C^{2b+\alpha}$, $0<\alpha<1$, then there exists a matrix $G(t,x;\tau,\xi)$, defined in $\overline{Q}\times\overline{Q}$ for $t>\tau$, having $2b$ derivatives with respect to x and one derivative with respect to t, and possessing the following properties:

* Everywhere the algebraic Lopatinskii condition is assumed to be satisfied ([1–4]).

a) $G(t,x;\tau,\xi) = Z(t,x;\tau,\xi) + v(t,x;\tau,\xi)$, where $Z(t,x;\tau,\xi)$ is the fundamental matrix of solutions of the system (1), (2); $G(t,x;\tau,\xi)$, for $t>\tau$, satisfies in t and x the system $Lu=0$ and the boundary conditions (3), and $v(t,x;\tau,\xi)\big|_{t=\tau}=0$, when at least one of the points x or ξ lies inside the domain Ω.

b) If $f(t,x)\in C_{x,t}^{\alpha,0}(Q)$, $\varphi(x)\in C^\alpha(\Omega)$, then the vector-function

$$u(t,x) = \int_\Omega G(t,x;0,\xi)\varphi(\xi)\,d\xi + \int_0^t d\tau\int_\Omega G(t,x;\tau,\xi)f(\tau,\xi)\,d\xi$$

is a solution of the problem (1), (2), (3).

c) For arbitrary matrices $G(t,x;\tau,\xi)$ and $v(t,x;\tau,\xi)$ the estimates

$$\left|\frac{\partial^{m_0}}{\partial t^{m_0}}D_x^{m}G(t,x;\tau,\xi)\right| \le C_{m_0 m}(t-\tau)^{-(n+|m|+2bm_0)/2b}\exp\left\{-c\frac{|x-\xi|^q}{(t-\tau)^{1/(2b-1)}}\right\}, \tag{4}$$

$$\left|\frac{\partial^{m_0}}{\partial t^{m_0}}D_x^{m}v(t,x;\tau,\xi)\right| \le C_{m_0 m}(t-\tau)^{-(n+|m|+2bm_0)/2b}\times$$

$$\times\exp\left\{-c\frac{\left[|x-\xi|+\rho(\xi,S)\right]^q}{(t-\tau)^{1/(2b-1)}}\right\}$$

$$(2bm_0 + |m| \leq 2b);$$

$$\left|\Delta_x \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^{kG}(t,x;\tau,\xi)\right| \leq C_{k_0 k}|x-x'|^{\alpha}(t-\tau)^{-(n+2bk_0+|k|+\alpha)/2b} \times$$

$$\times \left[\exp\left\{-c\frac{|x-\xi|^q}{(t-\tau)^{1/(2b-1)}}\right\} + \exp\left\{-c\frac{|x'-\xi|^q}{(t-\tau)^{1/(2b-1)}}\right\}\right] \quad (5)$$

$$(2bk_0 + |k| = 2b);$$

$$\left|\Delta_t \frac{\partial^{s_0}}{\partial t^{s_0}} D_x^{sG}(t,x;\tau,\xi)\right| \leq C_{s_0 s}|t-t'|^{(2b-2bs_0-|s|+\alpha)/2b} \times$$

$$\times \left[(t-\tau)^{-(n+2b+\alpha)/2b} \exp\left\{-c\frac{|x-\xi|^q}{(t-\tau)^{1/(2b-1)}}\right\} + \right.$$

$$\left. +(t'-\tau)^{-(n+2b+\alpha)/2b} \exp\left\{-c\frac{|x-\xi|^q}{(t'-\tau)^{1/(2b-1)}}\right\}\right] \quad (6)$$

$$(0 < 2bs_0 + |s| \leq 2b)$$

and the same estimates hold for the derivatives of $v(t,x;\tau,\xi)$, only everywhere, instead of $|x-\xi|$ and $|x'-\xi|$, one must put, respectively, $|x-\xi| + \rho(\xi,S)$ and $|x'-\xi|+\rho(\xi,S)$. In the estimates (4)–(6), $q = 2b/(2b-1)$, $\rho(\xi,S)$ is the distance of the point ξ to the boundary S, $\Delta_x f(x) = f(x) - f(x')$, $\Delta_t f(t) = f(t) - f(t')$. The constants $C_{m_0 m}$, $C_{k_0 k}$, $C_{s_0 s}$, and c depend on the norms of the coefficients of the system and of the boundary operators, on various characteristics of the boundary S and of the domain Ω, on the numbers $2b$, n, α, T, and δ from the parabolicity condition.

d) If the coefficients of the system and of the boundary operators do not depend on t, then for $G(t,x;\tau,\xi) \equiv G(t-\tau,x,\xi)$ and $v(t,x;\tau,\xi) \equiv v(t-\tau,x,\xi)$ the estimates (4)–(6) hold, in which the constants $C_{m_0 m}$, $C_{k_0 k}$, and $C_{s_0 s}$ are replaced respectively by the functions $C'_{m_0 m}e^{A(t-\tau)}$, $C'_{k_0 k}e^{A(t-\tau)}$, $C'_{s_0 s}e^{A(t-\tau)}$, where $C'_{m_0 m}$, $C'_{k_0 k}$, $C'_{s_0 s}$, and also c, do not depend on T, and A is some nonnegative number.

2) If one assumes that

$$A_k(t,x) \in C^{l+\alpha,\ \frac{1}{2b}(l+\alpha)}_{x,t}(Q), \qquad b^{(k)}_{vj}(t,x) \in C^{2b-r_v+l+\alpha,\ \frac{1}{2b}(2b-r_v+l+\alpha)}_{x,t}(\Gamma), \quad S \in C^{2b+l+\alpha}, \quad l > 0,$$

then for $G(t,x;\tau,\xi)$ and

for $v(t,x;\tau,\xi)$ the estimates (4)–(6) are valid, in which $2bm_0 + |m| \le 2b + l$, $2bk_0 + |k| = 2b + l$, $l < 2bs_0 + |s| \le 2b + l$, and in the estimate (6), instead of $2b - 2bs_0 - |s|$, $n + 2b$ there stand $2b + l - 2bs_0 - |s|$, $n + 2b + l$.

The proof is based on an analysis of an explicit, very cumbersome formula by means of which the solution of problem (1), (2), (3) is given ([3,4]), on estimates of integrals of the type of volume and surface parabolic potentials, in which there are systematically used sharp estimates of the fundamental matrices of solutions of systems of equations, previously obtained by the authors, sharp estimates of the Green matrices of homogeneous parabolic boundary problems in the half-space, sharp estimates of half-space fundamental matrices of solutions, and the equality to zero of the averages of their corresponding derivatives ([2,4]).

2. From Theorem 1, by the usual method ([2]), one can obtain sharp results on the Green matrix of the homogeneous elliptic boundary problem

$$\sum_{|k|\le 2b} A_k(x) D^k_x u + \lambda u = f(x), \tag{7}$$

$$B_\nu(x, D_x) u\big|_S = 0, \qquad \nu = 1, 2, \dots, bN, \tag{8}$$

$\operatorname{Re}\lambda > A$ (A is the number discussed in item) of Theorem 1), generated by the parabolic boundary problem (1), (2), (3).

Theorem 2. 1) If $A_k(x) \in C^\alpha(\Omega)$, $b^{(k)}_{\nu j}(x) \in C^{2b-r_j+\alpha}(\Gamma)$ and $S \in C^{2b-\alpha}$, then there exists a matrix $\Phi(x,\xi;\lambda)$, defined in $\overline{\Omega} \times \overline{\Omega}$ for $x \ne \xi$, having $2b$ derivatives with respect to x and possessing the following properties:

a)

$$\Phi(x,\xi;\lambda) = \int_0^\infty G(t,x,\xi;\lambda)\,dt = \varphi(x,\xi;\lambda) + w(x,\xi;\lambda),$$

where $G(t,x;\xi,\lambda)$ is the Green matrix of the parabolic problem generating problem (7), (8); $\varphi(x,\xi;\lambda)$ is the fundamental matrix of solutions of system (7); $\Phi(x,\xi;\lambda)$ for $x \ne \xi$ satisfies the homogeneous system (7) and the boundary conditions (8).

b) If $f(x) \in C^\alpha(\Omega)$, then the vector-function

$$u(x) = \int_\Omega \Phi(x,\xi;\lambda) f(\xi)\, d\xi$$

is a solution of problem (7), (8).

c) For the matrix $\Phi(x,\xi;\lambda)$ the estimates

$$|D_x^m \Phi(x,\xi;\lambda)| \le C_m e^{-c_0\delta|x-\xi|} \begin{cases} C_1, & n+|m|<2b, \\ C_1 \ln \dfrac{1}{|x-\xi|} + C_2, & n+|m|=2b, \\ C_1 |x-\xi|^{-n-|m|+2b}, & n+|m|>2b \end{cases} \quad (9)$$

$$(|m| \le 2b),$$

$$\left|\Delta_x D_x^k \Phi(x,\xi;\lambda)\right| \le$$

$$\le C_k |x-x'|^\alpha \left[e^{-c_0\delta|x-\xi|} |x-\xi|^{-n-|k|+2b-\alpha} + e^{-c_0\delta|x'-\xi|} |x'-\xi|^{-n-|k|+2b-\alpha} \right],$$

$$(|k| = 2b), \quad (10)$$

where $\delta = (\operatorname{Re}\lambda - A)^{1/2b}$, and the constants depend on the norms of the coefficients of the system and of the boundary operators, on various characteristics of the boundary S and the domain Ω, and on the numbers $2b, n, \alpha, \delta$. For the matrix $w(x,\xi;\lambda)$ the estimates (9), (10) hold, in which $|x-\xi|$ and $|x'-\xi|$ are replaced respectively by $|x-\xi| + \rho(\xi,S)$ and $|x'-\xi| + \rho(\xi,S)$.

[2]) If $A_k(x) \in C^{l+\alpha}(\Omega)$, $b_{vj}^{(k)}(x) \in C^{2b-r_v+l+\alpha}(\Gamma)$, and $S \in C^{2b+l+\alpha}$, $l>0$, then for $\Phi(x,\xi;\lambda)$ the estimates (9), (10) are valid, in which $|m| \le 2b+l$, $|k| = 2b+l$.

Let us note that the results obtained imply the solvability of the boundary-value problems (1), (2), (3) and (7), (8) under restrictions on $f(t,x)$, $\varphi(x)$, and $f(x)$ weaker than those formulated in Theorems 1 and 2: it is sufficient to assume that these functions satisfy the Dini condition.

3. The results obtained above make it possible, using the usual techniques ([2]), to obtain sharp estimates also for the Green functions (kernels) $\Phi_\beta(x,\xi;\lambda)$ of fractional negative powers of the elliptic operator corresponding to the problem (7), (8), defined by the formulas

$$\Phi_\beta(x,\xi;\lambda) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} G(t,x,\xi;\lambda)\,dt, \qquad 0<\beta<1.$$

Theorem 3. If the conditions of the first part of Theorem 2 are satisfied, then $\Phi_\beta(x,\xi;\lambda)$ is defined in $\overline{\Omega}\times\overline{\Omega}$ for $x\neq\xi$, has derivatives with respect to x up to order $2b$, for which the estimates

$$\left|D_x^m \Phi_\beta(x,\xi;\lambda)\right| \le C_m e^{-c_0\delta|\xi-x|} \times \begin{cases} C_1, & n+|m|<2b\beta, \\ C_1 \ln\dfrac{1}{|x-\xi|} + C_2, & n+|m|=2b\beta, \\ C_1|x-\xi|^{-n-|m|+2b\beta}, & n+|m|>2b\beta. \end{cases}$$

are valid.

Estimates of the Green function of boundary-value problems for a divergent parabolic equation of second order were established in ([6]); those for an elliptic boundary-value problem were announced in ([7]).

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Note: Figure translations are in progress. See original paper for figures.