OPTIMAL TRANSFERS BETWEEN CLOSE NEAR-CIRCULAR NON-COPLANAR ORBITS

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Abstract

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OPTIMAL TRANSFERS BETWEEN CLOSE NEAR-CIRCULAR NON-COPLANAR ORBITS

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The problem of transfers between orbits is considered by means of the linearized theory (^{1}). For this theory to be applicable, the initial and final orbits must be located in a small neighborhood of a circular orbit of radius r_{cp}.

The motion is considered in the cylindrical coordinate system $Or\varphi z$, whose plane $Or\varphi$ coincides with the plane of the initial orbit (see Fig. 1). The angle φ is measured from the line of nodes. The equations of the initial and final orbits are given in the form

$$r_0(\varphi) = p_0/[1 + e_0 \cos(\varphi - \varphi_{\pi,0})]$$

$$z_0(\varphi) = 0; \tag{1}$$

$$r_{N+1}(\varphi) = p_{N+1}/[1 + e_{N+1} \cos(\varphi -$$

$$-\varphi_{\pi,N+1})], \quad z_{N+1}(\varphi) = r_{\text{cp}} \Delta i \sin\varphi,$$

where Δi is the angle between the planes of the orbits.

Fig. 1. 1—the initial orbit, 2—the final orbit

The transition between the orbits is accomplished by means of $N+1$ impulses applied at $\varphi = \varphi_k$ and having, respectively, radial, transverse, and lateral components Δv_{rk}, $\Delta v_{\tau k}$, and Δv_{zk} $(k = 0, 1, \ldots, N)$. The linearized relations describing the orbit obtained after application of all the impulses are written in the form:

$$\bar{r}_{N+1}(\varphi) = \bar{r}_0(\varphi) + \sum_{k=0}^{N} \{2\Delta\bar{v}_{\tau k}[1 - \cos(\varphi - \varphi_k)] + \Delta\bar{v}_{rk}\sin(\varphi - \varphi_k)\},$$

$$\bar{z}_{N+1}(\varphi) = \bar{z}_0(\varphi) + \sum_{k=0}^{N} \Delta\bar{v}_{zk}\sin(\varphi - \varphi_k),$$

$$\bar{t}_{N+1}(\varphi) = \bar{t}_0(\varphi) + \frac{1}{2\pi}\sum_{k=0}^{N} \{\Delta\bar{v}_{\tau k}[3(\varphi - \varphi_k) - 4\sin(\varphi - \varphi_k)] + \tag{2}$$

$$+ 2\Delta\bar{v}_{rk}[1 - \cos(\varphi - \varphi_k)]\},$$

$$\bar{v}_{r,N+1}(\varphi) = \bar{v}_{r,0}(\varphi) + \sum_{k=0}^{N} \{2\Delta\bar{v}_{\tau k}\sin(\varphi - \varphi_k) + \Delta\bar{v}_{rk}\cos(\varphi - \varphi_k)\},$$

$$\bar{v}_{\tau,N+1}(\varphi) = \bar{v}_{\tau,0}(\varphi) + \sum_{k=0}^{N} \left\{2\Delta\bar{v}_{\tau k}\left[\cos(\varphi - \varphi_k) - \frac{1}{2}\right] - \Delta\bar{v}_{rk}\sin(\varphi - \varphi_k)\right\},$$

$$\bar{v}_{z,N+1}(\varphi) = \bar{v}_{z,0}(\varphi) + \sum_{k=0}^{N} \Delta\bar{v}_{zk}\cos(\varphi - \varphi_k).$$

Bars denote dimensionless quantities. The linear quantities r and z are referred to r_{cp}. The velocity components v_r, v_τ, v_z and the impulse components are referred to the velocity on a circular orbit with $r = r_{\text{cp}}$, while the time t is referred to the time of one complete revolution on such an orbit. Equations (2) are the starting point for solving various transfer problems in the multi-impulse formulation. Below we shall present the results of solving the problem of transfers between orbits optimal with respect to the magnitude of the characteristic velocity

$$\Delta\bar{v}_\Sigma = \sum_{k=0}^{N} \sqrt{\Delta\bar{v}_{rk}^2 + \Delta\bar{v}_{\tau k}^2 + \Delta\bar{v}_{zk}^2} \tag{3}$$

The impulse parameters $\varphi_k, \Delta\bar{v}_{rk}, \Delta\bar{v}_{\tau k}$, and $\Delta\bar{v}_{zk}$ $(k = 0, 1, \dots, N)$, as well as the number N, are varied. From these equations it is seen that the solution of this problem depends on the differences (see Fig. 1)

$$\begin{aligned} \Delta\bar{r}(\varphi) &= \bar{r}_{N+1}(\varphi) - \bar{r}_0(\varphi) = \Delta_0 + \Delta_c \cos\varphi + \Delta_s \sin\varphi, \\ \Delta\bar{z}(\varphi) &= \bar{z}_{N+1}(\varphi) - \bar{z}_0(\varphi) = \Delta_z \sin\varphi, \end{aligned} \tag{4}$$

where

$$\Delta_0 = (p_{N+1} - p_0)/r_{\text{cp}}, \quad \Delta_c = e_0 \cos\varphi_{\pi,0} - e_{N+1}\cos\varphi_{\pi,N+1};$$

$$\Delta_s = e_0 \sin\varphi_{\pi,0} - e_{N+1}\sin\varphi_{\pi,N+1}, \quad \Delta_z = \Delta i.$$

Along with the parameters $\Delta_0, \Delta_c, \Delta_s$, and Δ_z, the related parameters $\chi, \varphi_{\max}$, and σ are used:

$$\chi = \Delta_0/\sqrt{\Delta_c^2 + \Delta_s^2}; \quad \cos\varphi_{\max} = \Delta_c/\sqrt{\Delta_c^2 + \Delta_s^2}; \quad \sigma = \sqrt{\Delta_c^2 + \Delta_s^2}/\Delta_z. \tag{5}$$

The first two equations of system (2), taking (4) into account, are equivalent to the following equations:

$$\Delta_0 = \sum_{k=0}^{N} 2\Delta\bar{v}_{\tau k} \quad \text{(a)}, \qquad -\Delta_c = \sum_{k=0}^{N} (2\Delta\bar{v}_{\tau k}\cos\varphi_k + \Delta\bar{v}_{rk}\sin\varphi_k) \quad (), \tag{6}$$

$$\Delta_s = \sum_{k=0}^{N} (-2\Delta\bar{v}_{\tau k}\sin\varphi_k + \Delta\bar{v}_{rk}\cos\varphi_k) \quad (\ ), \qquad 0 = \sum_{k=0}^{N} \Delta\bar{v}_{zk}\sin\varphi_k \quad (\ ),$$

$$\Delta_z = \sum_{k=0}^{N} \Delta\bar{v}_{zk}\cos\varphi_k \quad (\ ). \tag{6}$$

These equations are constraints imposed on the varied parameters $\varphi_k, \Delta\bar{v}_{rk}, \Delta\bar{v}_{\tau k}$, and $\Delta\bar{v}_{zk}$.

The Lagrange function is written in the form

$$H = \sum_{k=0}^{N} \sqrt{\Delta\bar{v}_{rk}^2 + \Delta\bar{v}_{\tau k}^2 + \bar{v}_{zk}^2} + \lambda_1\text{(a)} + \lambda_2() + \lambda_3(\ ) + \lambda_4(\ ) + \lambda_5(\ ),$$

where $\lambda_1, \dots, \lambda_5$ are Lagrange multipliers. After differentiating with respect to all varied parameters and setting the derivatives equal to zero, the following equations are obtained:

$$\begin{aligned}
&\Delta\bar{v}_{rk}/\Delta\bar{v}_k + \lambda_2 \sin\varphi_k + \lambda_3 \cos\varphi_k = 0,\\
&\Delta\bar{v}_{\tau k}/\Delta\bar{v}_k + 2(\lambda_1 + \lambda_2 \cos\varphi_k - \lambda_3 \sin\varphi_k) = 0,\\
&\Delta\bar{v}_{zk}/\Delta\bar{v}_k + \lambda_4 \sin\varphi_k + \lambda_5 \cos\varphi_k = 0, \qquad \Delta\bar{v}_{rk}^2 + \Delta\bar{v}_{\tau k}^2 + \Delta\bar{v}_{zk}^2 = \Delta\bar{v}_k^2,\\
&\Delta\bar{v}_{rk}(\lambda_2 \cos\varphi_k - \lambda_3 \sin\varphi_k) - 2\Delta\bar{v}_{\tau k}(\lambda_2 \sin\varphi_k + \lambda_3 \cos\varphi_k)\\
&\qquad + \Delta\bar{v}_{zk}(\lambda_4 \cos\varphi_k + \lambda_5 \sin\varphi_k) = 0 \quad (k = 0, 1, \dots, N).
\end{aligned} \tag{7}$$

All solutions of the system of equations (6)–(7) have been found and investigated, each of them corresponding to different local extrema. For each of them, the regions of existence have been determined and $\Delta\bar{v}_\Sigma$ has been computed.

It has been established that, depending on the mutual position of the orbits, the absolute minimum of the characteristic velocity is attained for one of three types of transfers: either for transfers with impulses at the nodes (type I), or for transfers with impulses on one side of the line of nodes (type II), or for the so-called degenerate transfers (type III). It is proved that these transfers can be realized by means of two impulses. In the particular case when the orbits have a point of intersection, transfers of types I and II can be realized by means of an impulse applied at this point. Increasing the number of impulses does not lead to a decrease in the energetics of the transfers. The existence of families of multi-impulse isoenergetic transfers with energy equal to that of two-impulse transfers has been established. We give formulas for calculating each of the indicated transfers for the case of two impulses, when

$$\Delta_0 \geq 0, \quad \Delta_c \geq 0, \quad \Delta_s \geq 0, \quad \Delta_z \geq 0; \quad (\chi \geq 0, \quad \sigma \geq 0, \quad 0 \leq \varphi_{\max} < \pi/2). \tag{8}$$

Transfers with impulses at the nodes.

$$\Delta\bar{v}_\Sigma = \frac{1}{2}\sqrt{\Delta_c^2 + 4(\Delta_s^2 + \Delta_z^2)};$$

$$\varphi_1 = 0, \quad \Delta\bar{v}_1/\Delta\bar{v}_\Sigma = \frac{1}{2}(1 - \Delta_0/\Delta_c), \quad \Delta\bar{v}_{r,1}/\Delta\bar{v}_1 = \Delta_s/\Delta\bar{v}_\Sigma,$$

$$\Delta\bar{v}_{\tau,1}/\Delta\bar{v}_1 = -\Delta_c/2\Delta\bar{v}_\Sigma, \quad \Delta\bar{v}_{z,1}/\Delta\bar{v}_1 = \Delta_z/\Delta\bar{v}_\Sigma; \tag{9}$$

$$\varphi_2 = 180^\circ, \quad \Delta\bar{v}_2/\Delta\bar{v}_\Sigma = \frac{1}{2}(1 + \Delta_0/\Delta_c), \quad \Delta\bar{v}_{r,2}/\Delta\bar{v}_2 = -\Delta_s/\Delta\bar{v}_\Sigma,$$

$$\Delta\bar{v}_{\tau,2}/\Delta\bar{v}_2 = \Delta_c/2\Delta\bar{v}_\Sigma, \quad \Delta\bar{v}_{z,2}/\Delta\bar{v}_2 = -\Delta_z/\Delta\bar{v}_\Sigma.$$

The transfers exist for $\chi \leq \cos\varphi_{\max}$.

Transfers with impulses on one side of the line of nodes. The calculation is performed successively by rows:

$$a = \frac{1}{2\sin\varphi_{\max}}\left[\frac{1}{\sigma} - \sigma(1-\chi^2)\right], \qquad \frac{\eta}{\nu} = a + \sqrt{a^2+1},$$

$$Y = \frac{-2\chi\sigma}{\sqrt{\sigma^2 + 2\sigma(\eta/\nu)\sin\varphi_{\max} + (\eta/\nu)^2}},$$

$$\sin\delta = -Y\cos\varphi_{\max}/2\chi, \quad \cos\delta \leq 0, \quad \sin(\varphi_1 - \delta) = -Y/2, \quad \cos(\varphi_1 - \delta) \leq 0,$$

$$\sin(\varphi_2 - \delta) = -Y/2, \quad \cos(\varphi_2 - \delta) \geq 0,$$

$$1/\nu = \sqrt{[1 + (\eta/\nu)^2]\{1 + (Y/4)^2[(\eta/\nu)^2 - 3]\}},$$

$$\Delta\bar{v}_\Sigma = -2\Delta_0/\nu Y[1 + (\eta/\nu)^2], \qquad K = \frac{-\sigma\cos\varphi_{\max}}{(\sigma\sin\varphi_{\max} + \eta/\nu)}\sqrt{\left(\frac{2}{Y}\right)^2 - 1}, \quad (10)$$

$$\Delta\bar{v}_1/\Delta\bar{v}_\Sigma = (1+K)/2, \quad \Delta\bar{v}_{r,1}/\Delta\bar{v}_1 = \nu\sqrt{1 - Y^2/4},$$

$$\Delta\bar{v}_{\tau,1}/\Delta\bar{v}_1 = \Delta_0/2\Delta\bar{v}_\Sigma, \quad \Delta\bar{v}_{z,1}/\Delta\bar{v}_1 = \nu\sqrt{1 - Y^2/4},$$

$$\Delta\bar{v}_2/\Delta\bar{v}_\Sigma = (1-K)/2, \quad \Delta\bar{v}_{r,2}/\Delta\bar{v}_2 = -\Delta\bar{v}_{r,1}/\Delta\bar{v}_1,$$

$$\Delta\bar{v}_{\tau,2}/\Delta\bar{v}_2 = \Delta\bar{v}_{\tau,1}/\Delta\bar{v}_1, \quad \Delta\bar{v}_{z,2}/\Delta\bar{v}_2 = -\Delta\bar{v}_{z,1}/\Delta\bar{v}_1.$$

Here η, ν, δ, Y are parameters expressed through the Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

The transfers exist under the condition $\cos\varphi_{\max} \leq \chi$.

Degenerate transfers

$$\Delta\bar{v}_\Sigma = \frac{1}{2}\sqrt{\Delta_c^2 + (\Delta_s + \sqrt{3}\,\Delta_z)^2},$$

$$m = \sqrt{\sigma^2 + 2\sqrt{3}\sigma\sin\varphi_{\max} + 3},$$

$$\sin\delta = \sigma\cos\varphi_{\max}/m, \qquad \cos\delta \le 0,$$

$$\tilde{\alpha} = -8\cos\delta/\sqrt{3}m - 1,$$

$$\tilde{\beta} = 8\sin\delta/\sqrt{3}m. \tag{11}$$

To determine φ_1, the equation obtained was

$$\Delta\tilde{v}_1\sin u_1 + \sqrt{\frac{\Delta\tilde{v}_1\Delta\tilde{v}_2}{1-\tilde{\alpha}^2-\tilde{\beta}^2}}\,[\tilde{\beta}\sin u_1 -$$

$$-(1-\tilde{\alpha})\cos u_1] = \frac{\sigma}{m}\,\chi, \tag{12}$$

where

$$u_1 = \varphi_1 - \delta, \quad \Delta\tilde{v}_1 = (1-\tilde{\alpha}^2-\tilde{\beta}^2)/$$

$$/\,2(1-\tilde{\alpha}\cos 2u_1 - \tilde{\beta}\sin 2u_1),$$

$$\Delta\tilde{v}_2 = 1 - \Delta\tilde{v}_1, \quad u_2 = \varphi_2 - \delta.$$

This equation has two roots. Consequently, there exist two isoenergetic two-impulse singular transfers. For a known φ_1, the subsequent calculations are carried out by the formulas

$$\sin u_2 = \sqrt{\Delta\tilde{v}_1}\,[\tilde{\beta}\sin u_1 -$$

$$-(1-\tilde{\alpha})\cos u_1]/\sqrt{\Delta\tilde{v}_2(1-\tilde{\alpha}^2-\tilde{\beta}^2)},$$

$$\operatorname{sign}[\sin 2u_2] = \operatorname{sign}(\tilde{\beta} - \Delta\tilde{v}_1\sin 2u_1),$$

$$\Delta v_k/\Delta v_\Sigma = \Delta\tilde{v}_k, \quad \Delta v_{rk}/\Delta v_k =$$

$$= -\cos u_k/2, \quad \Delta v_{\tau k}/\Delta v_k = \sin u_k,$$

$$\Delta v_{zk}/\Delta v_k = -\tfrac{1}{2}\sqrt{3}\cos u_k \quad (k = 1, 2).$$

Fig. 2. I, II, III—regions in which the absolute minimum Δv_Σ is attained, respectively, for transfers with impulses at the nodes, for transfers with impulses on one side of the line of nodes, and for singular transfers

The transfer exists under the conditions

$$1/\sqrt{3}\sigma \leq \sin\varphi_{\max}, \quad \chi \leq \sqrt{1 + 2\sin\varphi_{\max}/\sqrt{3}\sigma - 1/\sigma^2}. \tag{14}$$

If conditions (8) are not satisfied, then the transfer parameters are determined by a simple recalculation of the parameters determined by formulas (9)–(13). When the sign of Δ_0 changes, it is necessary to change the signs of Δv_{rk}, $\Delta v_{\tau k}$, and Δv_{zk}, and to shift the instants of application of the impulses by 180°. When the sign of Δ_c changes, it is necessary to change the signs of Δv_{rk} and Δv_{zk} and to move the impulses to points symmetric with respect to the normal to the line of nodes. When the sign of Δ_s changes, the sign of Δv_{rk} changes and the impulses are moved to points symmetric with respect to the line of nodes.

The regions of existence of transfers are shown in Fig. 2 in polar coordinates $O\chi\varphi_{\max}$ for various values of σ. The regions of existence of transfers of types I and II do not depend on σ and mutually complement one another. The region of existence of singular transfers decreases as σ decreases and disappears at $\sigma = 1/\sqrt{3}$. It intersects the regions of existence of transfers of types I and II. It has been shown that if a singular transfer exists, it is always more economical than the other types of transfer. In particular, the one-impulse transfer with the impulse at the node, which exists in the case of intersecting orbits, is the most economical only in those cases in which no two-impulse singular transfer exists.

In the course of analyzing extremal transfers, the existence was established of spatial extremal transfers that do not pass into extremal coplanar transfers [1] as $\Delta i \to 0$. In particular, the type-I transfer is such a transfer. However, as $\Delta i \to 0$ such transfers give a local extremum of Δv_Σ. The transfers that give the absolute minimum of Δv_Σ as $\Delta i \to 0$ pass into the most economical coplanar transfers.

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REFERENCES

1. G. E. Kuzmak, *Cosmic Research*, **3**, no. 1 (1965).

Note: Figure translations are in progress. See original paper for figures.