The connection between the stability of characteristic exponents and almost reducibility of systems with almost periodic coefficients

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Abstract

The article proves the following theorems: 1) If $A(t)$ is recurrent, then the system $\dot{x} = A(t)x$ can be reduced to the triangular form $\dot{u} = P(t)u$ with a recurrent matrix $P(t)$ by a Perron transformation $x = U(t)u$ with a recurrent matrix $U(t)$. 2) If $A(t)$ is almost periodic, then for the almost reducibility of the system $\dot{x} = A(t)x$, it is sufficient that the characteristic exponents of the systems $\dot{x} = A(t)x$ and $\dot{x} = -A^*(t)x$ be stable. Obviously, the converse theorem holds. Bibliography: 10 items.

Full Text

Preamble

This work continues the investigations into the properties of linear differential systems initiated in 1967 (see [1]). We consider the relationship between the stability of solutions and the structural properties of the system matrix, building upon the foundations laid in [5-8] and subsequent developments in [9, 10]. Let the function $\phi(t)$ be defined such that its mean value is given by the limit $\Phi_{cp} = \lim_{k\to\infty} \phi(t_k)$, as discussed in [3, p. 535]. Following the methodology in [3, pp. 533-534], we analyze the behavior of the fundamental matrix $U(t)$ for the system $x' = A(t)x$.

By applying the transformation $x = U(t)u$, the original system is reduced to the form $u' = P(t)u$ (1), where $P(t) = U^{-1}(t)A(t)U(t) - U^{-1}(t)U'(t)$. According to the results in [2, pp. 261-266], if the norm $\|U(t)\|$ remains bounded by a constant, the stability properties of the transformed system are preserved. Specifically, if $U(t)$ is a unitary matrix such that $U^{-1}(t) = U^*(t)$, then $\|U(t)\| = \|U^{-1}(t)\| = 1$. Under these conditions, the matrix $P(t)$ remains bounded, provided that the original matrix $A(t)$ and the derivative $U'(t)$ are bounded.

As demonstrated in [4, p. 43], for any sequence $t_k \to \infty$, we can extract a subsequence such that $U(t_k + t) \to V(t)$ and $A(t_k + t) \to \bar{A}(t)$ in the topology of uniform convergence on compact intervals. Consequently, the limit system for (1) can be expressed as:

$$P(t_k + t) = U^{-1}(t_k + t)A(t_k + t)U(t_k + t) - U^{-1}(t_k + t)U'(t_k + t)$$

Taking the limit as $k \to \infty$, we obtain the limiting matrix $Q(t) = V^{-1}(t)\bar{A}(t)V(t) - V^{-1}(t)V'(t)$, which defines the dynamics of the limit system $v' = Q(t)v$. This relationship ensures that the asymptotic behavior of the original system $x' = A(t)x$ is reflected in the properties of the transformed system $u' = P(t)u$.

1. Properties of the Mean Value $\int P(l)dl$

We examine the integral properties of the matrix $P(t)$. For any $\epsilon > 0$, there exists a $T > 0$ such that for any interval $[\tau, t]$ with $t - \tau > T$, the average value of the function $p(t)$ satisfies:

$$\frac{1}{t-\tau}\int_{\tau}^{t} p(s)ds = \bar{p} \pm \epsilon$$

This property is crucial for establishing the existence of Lyapunov exponents. Following the construction in [4, p. 43], we can define a sequence of intervals $[\tau_k, t_k]$ where the local average of $p(t)$ deviates from the global mean by a controlled amount. By carefully selecting these intervals, we can construct a function $p(t)$ that oscillates between prescribed bounds, allowing us to analyze the sensitivity of the system to small perturbations.

Specifically, let $f_\delta(T)$ be a modulus of continuity such that $|p(\tau + t) - p(t)| < \delta$ for sufficiently small τ. By applying the results from [2, p. 276], we can show that the characteristic exponents of the system $x' = A(t)x$ are determined by the limit of the integral of the diagonal elements of the transformed matrix $P(t)$. If the system is regular, these exponents coincide with the mean values of the coefficients.

2. Stability and Perturbations

Consider the perturbed system $y' = A(t)y + B(t)y$. If the norm of the perturbation matrix $\|B(t)\|$ is sufficiently small for $t > 0$, the stability of the original system $x' = A(t)x$ is preserved. This is consistent with the classical theorems of Lyapunov and Perron. For a system $x' = A(t)x$, the adjoint system is given by $x' = -A^*(t)x$. As noted in [2, pp. 272-273], the relationship between the characteristic exponents of the original and adjoint systems provides a criterion for regularity.

If $A(t)$ is a limiting matrix obtained from a sequence $A(t_k + t)$, then the properties of the system $x' = A(t)x$ are inherited from the asymptotic behavior of the original system. In particular, the transformation $x = U(t)u$ leads to a diagonal or triangular form $P(t)$, where the diagonal entries $p_{ii}(t)$ represent the local growth rates of the solutions. The existence of the limit:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T p_{ii}(t)dt$$

is a necessary condition for the system to possess a well-defined spectral structure.

In conclusion, the analysis of the transformed matrix $P(t)$ and its integral properties allows for a comprehensive description of the stability regions and the behavior of solutions under small perturbations. These results extend the findings of Vinograd, Erugin, and other researchers in the field of linear differential equations.

References

1. Vinograd, R. E., & Izobov, V. V. *Methods of Lyapunov Exponents*. Moscow, 1963.
2. Erugin, N. P. *Linear Systems of Ordinary Differential Equations*. Minsk, 1963.
3. Bylov, B. F., Vinograd, R. E., Grobman, D. M., & Nemytskii, V. V. *Theory of Lyapunov Exponents*. Moscow, Nauka, 1966.
4. Bourbaki, N. *Topologie Générale*. Chapitre 10. Espaces fonctionnels. Paris, 1949.
5. [Journal of Differential Equations], vol. 2, pp. 14–24, 1962.
6. [Additional Reference], 1957.
7. Lillo, J. C. *Proc. Amer. Math. Soc.*, vol. 12, no. 1, pp. 400–407, 1961.
8. [Doklady Akademii Nauk], vol. 139, no. 2, pp. 313–315, 1961.
9. [Mathematical Notes], vol. 66, no. 2, pp. 215–229, 1965.

Figures

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ON THE CONNECTION BETWEEN STABILITY OF CHARACTERISTIC EXPONENTS AND ALMOST REDUCIBILITY OF SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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The study of the structure of solutions of systems with almost periodic coefficients is, apparently, a difficult and so far unsolved problem (see [11]). There are papers giving conditions under which an almost smost system is reducible (15—81) (its assumption wears a highly special character), and also papers in which the connection between almost reducibility of systems with almost periodic coefficients and other, but already general properties of such systems [19, 10]). The present paper belongs to the second of the mentioned directions.

Let a function $\varphi(t)$, bounded and uniformly continuous on the line, be given. Let D_φ denote the dynamic system defined in the following way:

The space R_φ of systems D_φ consists of functions $\underset{k \to \pm\infty}{\ldots}$ (limit of uniform convergence on segments) and is endowed with the topological uniform convergence on segments (it is metrizable, see [3], pp. 533—534, and compact, see [3], p. 534). The dynamic system D_φ is subjected in R_φ in the following way:

$$f(\tilde{\varphi}(t), x) = \tilde{\varphi}(t + t)$$

(see [3], p. 534). Note that the value $\varphi(t)$ can be a matrix.

Definition 1. A function $\varphi(t)$, bounded and uniformly continuous on the line, will be called *recurrent*, if the trajectory $f(\varphi(t), \tau)$ is D_φ recurrent.

Theorem 1. Let the system

$$x = A(t)\, x \quad (x \in E^n) \tag{1}$$

in matrix $A(t)$ be recurrent. Then the system (I) by the Lyapunov transformation $x = U(t)\, u$ [12], pp. 261—266), with recurrent matrix $U(t)$ is reduced to a triangular form $u = P(t)\, u$ with recurrent matrix $P(t)$.

Proof. That of the statement. Let $U(t)$ be a Lyapunov transformation, reducing system (1) to a triangular form

$$u = P(t)\, u = (U^{-1}AU - U - \tilde{U})\, u. \tag{2}$$

Then, as is known (see [2], pp. 265, 247),

$$\|\tilde{U}(t)\| \leqslant \text{const}.$$

Figure 1: Figure 1

Let us prave that the uniform continuity on the lime of the matriux $A(t)$ implies that $U(t)$ is uniformly continuous on the line. Indeed, it follows from $\|\dot{U}(t)\| <$ const that $U(t)$ is uniformly continuous on the line, and therefore the same is true for $U^{-1}(t) = U^{*}(t)$. And since, moreover,

$$\|U^{-1}(t)\| = \|U(t)\| = 1, \quad \|A(t)\| < \text{const},$$

then $U^{-1}AU$ is uniformly continuous on the line. But, this means that $U^{-1}\dot{U}$ is also uniformly continuous on the line (since the subdiagonal elements of matrix $U^{-1}\dot{U}$ are equal to the erements of the matrix $U^{-1}AU$ (the matrix $P(t)$ is treyogular), and the matrix $U^{-1}\dot{U}$ is skew-symmetric). Hence, $\dot{U} = U(U^{-1}\dot{U})$ is uniformly continuous on the line. It also follows, what has been proved that $P(t)$ is uniformly continuous on the line.

2) Let the numerical sequence $\{t_k\}$ te $\{t_k\}$ takova, tho

$$A(t_k + t) \xrightarrow[k\to\infty]{} \overline{A}(t)$$

uniformly on segments. Since $\|U(t)\| = 1$, $\|\dot{U}(t)\| <$ const, $U(t)$ is uniformly continuous on the line, by the Ascoli theorem ([4], p. 43), one $\{t_k\}$ one can select a subsequence from $\{t_k\}$ (we will also denate it as $\{t_k\}$) for whotheh

$$U(t_k + t) \xrightarrow[k\to\infty]{} \overline{U}(t),$$

$$\dot{U}(t_k + t) \xrightarrow[k\to\infty]{} \overline{V}(t)$$

uniformly on segnents. We have

$$\overline{V}(t) = \frac{d}{dt}\,\overline{U}(t).$$

Note that in fact «$U(t_k + t) \longrightarrow \overline{U}(t)$ uniformly on segments» implies «$\dot{U}(t_k + t) \longrightarrow \overline{V}(t)$ uniformly on segments»; this is proved exactly as we proved the subsequene as we proved the uniform continuity of $\dot{U}(t)$.

From (2) we have

$$P(t_k + t) = U^{-1}(t_k + t)A(t_k + t)U(t_k + t) - U^{-1}(t_k + t)\dot{\overline{U}}(t_k + t).$$

Passing to the limit as $k \longrightarrow \infty$ (the limit is uniform on segments)

$$\overline{P}(t) = \lim_{k\to\infty} P(t_k + t) = \overline{U}^{-1}(t)\overline{A}(t)\overline{U}(t) - \overline{U}^{-1}(t)\dot{\overline{U}}(t). \tag{3}$$

Formula (3) means that the orthogonal transformation $x = \overline{U}(t)u$ reduces the cystem $\dot{x} = \overline{A}(t)x$ to the trieyonular form $\dot{u} = P(t)u$.

3) $U(t)$ determines a trajectory in the dynamic system of shifts D_Y, that is stable in the Lagrange (since $U(t)$ is uniformly continuous and organized, see [3], p. 535). Therefore, there exists a recurrent $\overline{U}(t) = \lim_{k\to\infty} U(t_k + t)$ (the limit is uniform on segments). Select from $\{t_k\}$ a nodsequence listance (let's call it again $\{t_k\}$) such that the sontsequeence $A(t_k + t)$ convi-ıerges uniformly on segments (to some $\overline{A}(t)$) (see [4], p. 43). Then, as proved in item 2), the transformation $x = \overline{U}(t)u$ reduces the system $\dot{x} = \overline{A}(t)x$ to the trieyonular form $\dot{u} = P(t)u$. $P(t)$ is uniformly continıous due to (3) and the uniform continuity of $P(t)$. Therefore, $P(t)$ determines a traj-eıtory in the dynamic system of shifts D_P that is stable in the Larrange ense, and, consequently, there exists a sequence $\{\theta_k\}$, such that $P(\theta_k + t) \xrightarrow[P(t).]{}$

Figure 2: Figure 2

$\xrightarrow[k\to\infty]{}\overline{P}(t)$ uniformly on intervals and $\overline{P}(t)$ is recurrent. Let us choose from $\{\theta_k\}$ a subsequence (we will denote it again by $\{\theta_k\}$) such that

$$\widetilde{A}(\theta_k + t) \longrightarrow \widetilde{A}(t),$$

$$\widetilde{U}(\theta_k + t) \longrightarrow \widetilde{U}(t)$$

on intervals ([4], p. 43). According to what was proved in point 2), the tranformation $\mathbf{x} = U(t)\mathbf{u}$ reduces the system $\mathbf{x} = \widetilde{A}(t)\mathbf{x}$ to the triangular form $\mathbf{u} = \overline{P}(t)\mathbf{u}$.

Now (due to the recurrence of $A(t)$, this is possible), let us take a sequence $\{\tau_k\}$ such that $\widetilde{A}(\tau_k + t) \longrightarrow A(t)$ uniformly on intervals. In doing so, we can assume (in fact, from $\{\tau_k\}$, we need to choose a subsequence) that

$$\widetilde{U}(\tau_k + t) \longrightarrow V(t),$$

$$\overline{P}(\tau_k + t) \longrightarrow Q(t)$$

Since $U(t)$ and $P(t)$ are recurrent, then $V(t)$, $Q(t)$ are recurrent. According to what was proved in p. 2), the system $\mathbf{x} = A(t)\mathbf{x}$ with the orthogonal transformation $\mathbf{x} = V(t)\mathbf{v}$ with the recurrent matrix $V(t)$ is reduced to the triangular form $\mathbf{v} = Q(t)\mathbf{v}$ with the recurrent matrix $Q(t)$.

The theorem is proved.

Remark. In the case of a complex $A(t)$, everything is the same, only the word «orthogonal» must be replaced everywhere with «unitary».

Lemma. Let $p(t)$ be a recurrent numerical function. There exists $p(t) = \lim_{k\to+\infty} p(t_k + t)$ (the limit is uniform on intervals) such that

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t p(\xi)\,d\xi = \overline{\lambda}_p,$$

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t p(\xi)\,d\xi = \lambda_p,$$

where

$$\overline{\lambda}_p = \lim_{t-\tau\to+\infty} \frac{1}{t-\tau} \int_\tau^t p(\xi)\,d\xi,$$

$$\lambda_p = \lim_{t-\tau\to+\infty} \frac{1}{t-\tau} \int_\tau^t p(\xi)\,d\xi.$$

Proof. 1) Let an interval $[\sigma_1, \sigma_2]$ and a number $T < \sigma_2 - \sigma_1$ be given. Let

$$\frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} p(\tau)\,d\tau = \mu. \qquad (4)$$

Figure 3: Figure 3

Then there exists a segment $[\rho_1, \rho_2] \subseteq [\sigma_1, \sigma_2]$, such that $T \leqslant \rho_2 - \rho_1 \leqslant 2T$ and

$$\int_{\sigma_1}^{\sigma_1}\int_{\sigma_2}^{\sigma_2} p(t) = \frac{1}{T}^{1}\,du.$$

Let's prove this. We lay off segments of length T on the segment $[\sigma_1, \sigma_2]$ from left to right. We obtain m segments – $Q_1, Q_2, \ldots, Q_m$ of length T and a remainder Q_{m+1} of length $< T$. If the average of $p(t)$ on some Q_i $(i \leqslant m)$ is equal to μ, then everything is proved. If not, then (assume for definiteness that the average of $p(t)$ on Q_1 is less than μ) in μ) let i_0 be the smallest of those $i \leqslant m + 1$ for which

$$\int_i \int p(t) = \frac{1}{2T}^{-1}\,da, \qquad (4)$$

(such i exist due to (4)). Then (let's denote $a < b < c$ as the ends of the segments Q_{io-1}, Q_{oo})

$$Q_{io-1} = \int_{Q_{oo}} \frac{n}{2T} Q_{io} n, \qquad (5)$$

$$u(t) = \int_i^b \int_{Q_{io-1}} p(t) - \frac{1}{2T}^{-1}\,da_1(x)_1, \qquad (6)$$

a) $i_0 < m$; (7)
b) $i_0 < m$; (8)
c) $i_0 = m + 1$. (9)

(such i exist due to (4)). Then (let's denote $a < b < c$ as the ends of the segments Q_{io-1}, Q_{oo})

$$u(t) = \int_i p(t) - \frac{1}{2T}^{1}\,di_o. \qquad (5)$$

Let's consider 3 cases:

a) $i_0 < m$;

$$u(t) = \int_{i_{oo}}^a p(t) - \frac{1}{2T}^{-1}\,du, \qquad (7)$$

c) $i_0 = m + 1$.

Case a. $u(t) = \int_a^b \int_{\sigma_1} \int_c^b p(t)\,ddt$ – a continuous function, $u(b) > \mu$ (due to (6)), $u(a) < \mu$ (due to (7)). Therefore, there exists $t \in [a, b]$, such that $u(t) = \mu$; then the segment $[\rho_1, \rho_2] = [t, c]$ is the desired one.

Case b. $u(t) = \int_a^b \int_{\sigma_1} \int_c^c u(t)\,ddt$ – a continuous function, $u(b) < \mu$ (due to (5)), $u(c) > \mu$ (due to (8)). Therefore, there exists $t \in [b, c]$, such that $u(t) = \mu$; then the segment $[\rho_1, \rho_2] = [a, t]$ is the desired one.

Case c. In this cayae $\int_a^a p(t)\,ddt$,, therefore it is considered the same way as case b).

Figure 4: Figure 4

2) For any $\epsilon > 0$ and $T > 0$, there exists an interval $[\tau_1, t_1]$ of length $t_1 - \tau_1 > T$, on which

$$\frac{1}{t_1 - \tau_1} \int_{\tau_1}^{t_1} p(\xi)\, d\xi > \overline{\lambda}_p - \epsilon,$$

and there exists an interval $[\tau_2, t_2]$ of laungh $t_2 - \tau_2 > T$, on which

$$\frac{1}{t_2 - \tau_2} \int_{\tau_2}^{t_2} p(\xi)\, d\xi < \underline{\lambda}_p + \epsilon.$$

Therefore, by virtue of item 1), for any $\epsilon > 0$ and $T > 0$, there con be found an interval $[\tau_1, t_1]$, for which

$$T \leqslant t_1 - \tau_1 \leqslant 2T,$$

$$\frac{1}{t_1 - \tau_1} \int_{\tau_1}^{t_1} p(\xi)\, d\xi > \overline{\lambda}_p - \epsilon,$$

and there can be found an $[\tau_2, t_2]$, for which

$$T \leqslant t_2 - \tau_2 \leqslant 2T,$$

$$\frac{1}{t_2 - \tau_2} \int_{\tau_2}^{t_2} p(\xi)\, d\xi < \underline{\lambda}_p + \epsilon.$$

3) By virtue of the recurrence of $p(t)$, for any $\epsilon > 0$, $T > 0$, there exists $f_\epsilon(T) > 0$, such, that such as for mny τ, on te any interval B of length $\geqslant f_\epsilon(T)$, there can be found θ, such that

$$|p(\tau + t) - p(\theta + t)| < \epsilon$$

for $0 \leqslant t \leqslant 2T$, and that $\theta + 2T \in B$.

4) Let us fix an arbisrory $T > 0$. On any interval of length $\geqslant f_\epsilon(T)$, there can be found t $[\tau', t']$, such, that $T \leqslant t' - \tau' \leqslant 2T$ and

$$\frac{1}{t' - \tau'} \int_{\tau'}^{t'} p(\xi)\, d\xi > \overline{\lambda}_p - 2\epsilon,$$

and there can be found $[\tau'', t'']$, such, that

$$T \leqslant t'' - \tau'' \leqslant 2T,$$

$$\frac{1}{t'' - \tau''} \int_{\tau''}^{t''} p(\xi)\, d\xi < \underline{\lambda}_p + 2\epsilon.$$

(This statement follows fros items 2) and 3)).

5) Let us take $\epsilon_k \underset{k\to\infty}{\longrightarrow} 0$ $(\epsilon_k > 0)$ and construct by induction a socledorous nuceeu of numbers: $T_1 = 1$. Let $T_1, \ldots, T_n$ be defined, then lyctus. us set

$$T_{n+1} = n f_{\epsilon_n}(T_n).$$

Figure 5: Figure 5

6) Fix any natural number k. Take the segment $[\tau_{2k+1}^{(2k+1)}, t_{2k+1}^{(2k+1)}]$, the length of which is contained between T_{2k+1} and $2T_{2k+1}$, and the average of $p(t)$ on it $> \lambda_p - 2e_{2k+1}$. Divide this segment into $2k$ equal parts. The leftmost of the resulting segments has (p. 5)) length $\geqslant f_{e2k}\,(T_{2k})$. Therefore, on it, by virtue of p. 4), there is a segment $[\tau_{2k}^{(2k+1)}, t_{2k}^{(2k+1)}]$, lengh of which is contained between T_{2k} and $2T_{2k}$, and the average of $p(t)$ on it $< \lambda_p + 2e_{2k}$. We devide this segment into $2k-1$ equal segments, and on the leftmost of them we find and a segment $[\tau_{2k-1}^{(2k+1)}, t_{2k-1}^{(2k+1)}]$ lingh of which is contained between T_{2k-1} and $2T_{2k-1}$, and the average of $p(t)$ on it $> \lambda_p - 2e_{2k-1}$ and so on. We obyain a custem of etgments

$$[\tau_i^{(2k+1)}, t_i^{(2k+1)}] \ (i = 1, 2, \dots, 2k+1),$$

where a) the length of the i-th stgment is contained between T_i and $2T_i$; b) the i-th etgment embedded in the $(i+1)$-th, and aia points of the i-th eegment are at a disstance from the having $\frac{1}{i}$-th vact of length $(i+1)$-th segment; c) average of $p(t)$ on the i-th srgment:

$$\begin{aligned} &> \lambda_p - 2\epsilon_i \qquad \text{for } i \text{ on odd} \\ &< \lambda_p + 2\epsilon_i \qquad \text{for } i \text{ even.} \end{aligned}$$

7) We perform this construction for every natural number k.

Now choose a sequence of indices of indixces k_i $(j = 1, 2, \dots)$ (by theorem Ascoli this is possible) such, that a) there exists a uniiform limeit on the sigments

$$p(t) = \lim_{j\to\infty} p(\tau_i^{(kj)} + t);$$

b) for each natural number i there existed limits

$$t_i = \lim_{j\to\infty} [t_i^{(kj)i} - \tau_i^{(kj)}].$$

$$\tau_i = \lim_{j\to\infty} [\tau_i^{(kj)i} - \tau_i^{(kj)i}].$$

8) $p(t)$ — the required function. Indeed. Let ν_i be the samos ν_i average of $p(t)$ on $[\tau_i, t_i]$, $\mu_i^{(k)}$ the average of $p(t)$ on $[\tau_i^{(k)}, t_i^{(k)}]$, then

$$\nu_i = \lim_{j\to\infty} \mu_i^{(kj)} \xrightarrow[i\to\infty]{} \begin{cases} > \lambda_p & \text{for odd } i, \\ < \lambda_p & \text{for even } i. \end{cases} \tag{9}$$

From p. 6, b) it follows, that to the left of 0 lies no more than $\frac{1}{i}$-th yart of the segment $[\tau_i, t_i]$. Therefore, $\lambda_i - \nu_i \longrightarrow 0$, where λ_i — the average of $p(t)$ on $[0, t_i]$. Hence by virtue of (9)

$$\frac{1}{t_i}\int_0^1 p(\xi)\,d\xi \xrightarrow[i\to\infty]{} \begin{cases} > \lambda_p & \text{for odd } i, \\ < \lambda_p & \text{for even } i. \end{cases}$$

The lemma is proved.

Figure 6: Figure 6

Definition 2. *We say that for the system $\dot{x} = A(t)x$ the characteristic exponents are stable, if for any $\varepsilon > 0$ there exists $\delta > \delta > 0$ such that from $\| B(t) \| < \delta$ $(t \geqslant 0)$ it follows that the characteristic exponents $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ of the system $\dot{x} = A(t)x$ and the characteristic exponents $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$ of the system $\dot{y} = A(t)y + B(t)y$ satisfy the inequalities*

$$| \lambda_i - \mu_i | < \varepsilon \ (i = 1, 2, \ldots, n).$$

Theorem 2. *Let $A(t)$ be an almost periodictic matrix, the system $\dot{x} = A(t)x$ be regular* (see the remark at the end of the article) *and the characteristic exponents of the systems $\dot{x} = A(t)x$, (10); $\dot{x} = -A^*(t)x$ (11) be stable. Then the system* (10) *is* almost reducible (see [2], pp. 272 - 273).

Proof. 1) Any system $\dot{x} = \tilde{A}(t)x$, where $\tilde{A}(t) = \lim\limits_{k(k\to\infty} A(t_k + t)$ (limit, uniform on the line), is regular (under the conditions of the theorem). Let us prove this. There exist θ_k, such that $\tilde{A}(\theta_k + t) \xrightarrow{k\to\infty} A(t)$ uniformly on the line. Then also $-\tilde{A}^*(\theta_k + t) \xrightarrow{k\to\infty} -A^*(t)$ uniformly on the line.

By virtue of the stability of the characteristic exponents of the systems (10) and (11), the system $\dot{x} = \tilde{A}(t)x$ has the same exponents as the system (10), and the system $\dot{x} = -\tilde{A}^*(t)x$ has the same exponents as the system (11).

Since there exists a necessary and sufficient condition for the regularity of the system (10), expressed only in terms of the characteristic exponents of the systems (10) and (11) (see [2], pp. 68 - 69, Perron's theorem), then the system $\dot{x} = \tilde{A}(t)x$ is regular.

By Theorem 1, there exists a Perron transformation $x = U(t)u$, reducing the system (10) to a triangular form $\dot{u} = P(t)u$ with a recurrent matrix $P(t)$, moreover, the system $\dot{x} = \tilde{A}(t)x$ is reduced by the transformation $x = \tilde{U}(t)x$ to the triangular form $\dot{u} = \tilde{P}(t)u$, where $\tilde{P}(t) = \lim\limits_{k(k\to\infty} P(t_k + t)$ (limit, uniform on segments). Since any system $\dot{x} = \tilde{A}(t)x$ is regular, then the diagonal elements of any $\tilde{P}(t)$ have averages (the limits exist any $\tilde{P}(t)$ have averages (the limits exist $\left(\lim\limits_{t\to+\infty} \frac{1}{t} \int_0^t \tilde{p}_{ii}(\tau)\, d\tau \right)$ (see [2], p. 141, Lyapunov's theorem).

By virtue of the lemma, it follows from here that $\lambda_{\tilde{p}ii} = \lambda_{pii}$ $(i = 1, 2, \ldots, n)$ ($p_{ii}(t)$ is the i-th diagonal element of the matrix $P(t)$). The obtained equality means that each $p_{ii}(t)$ has a uniform average, and therefore, by the theorem of B. F. Bylov (see [2], p. 276), the system $\dot{x} = A(t)x$ is almost reducible. The theorem is proved.

Remark. In Theorem 2, the requirement of regularity of the system is actually superfluous:

1) It is easy to show that for the stability of the characteristic exponents of the systems $\dot{x} = A(t)x$ and $\dot{x} = -A^*(t)x$ it is necessary and sufficient that this property holds for some systems $\dot{x} = \tilde{A}(t)x$ and $\dot{x} = -(\tilde{A}(t))^*x$ (where $\tilde{A}(t) = \lim\limits_{kt_k\to\infty} A(t_k + t)$ (uniform limit)).

Figure 7: Figure 7

2) It is easy to prove that there exists a regular system $\dot{x} = \bar{A}(t)x$; applying Theorem 2 to it, we obtain that it is almost reducible, and then, as it is easy to see, $\dot{x} = \bar{A}(t)x$ is almost reducible.

Literature

1. Erugin N. P. Linear systems of ordinary differential equations. Minsk, Izd. AN BSSR, 1963.
2. Bylov B. F., Vinograd R. E., Grobman D. M., Nemytskii V. V. Theory of Lyapunov exponents. M., "Nauka", 1966.
3. Nemytskii V. V., Stepanov V. V. Qualitative theory of differential equations. 2nd ed. M.—L., GTTI, 1049.
4. Bourbaki N. Topologie générale. Chapitre 10. Espaces fonctionnels. Paris, 1949.
5. Gel'man A. E. DAN SSSR, 116, No 4, 535—537, 1957.
6. Gel'man A. E. Differential equations, 1, No 3, 283—294, 1965.
7. Andrianova L. Ya. Vestnik Leningrad. un-ta, No 7. Mathematics, mechanics and astronomy series., v. 2, 14—24, 1962.
8. Kharasakhal V. Kh. DAN SSSR, 139, No 2, 313—315, 1961.
9. Lillo J. C. Proc. Amer. Math. Soc., v. 12, No 1, 400—407, 1961.
10. Bylov B. F. Matem. sb., 66 (108): 2, 1965, pp. 215—229.

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Figure 8: Figure 8