Doklady of the Academy of Sciences of the USSR

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

Doklady of the Academy of Sciences of the USSR
1967, Volume 173, No. 2

513.88+517.948.35+517.948.5

MATHEMATICS

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ON THE MULTIPLICATIVE REPRESENTATION OF CHARACTERISTIC FUNCTIONS OF CONTRACTION OPERATORS

(Presented by Academician L. S. Pontryagin, 19 III 1966)

Let T be a linear contraction operator ($\|T\| \leq 1$), acting in a separable Hilbert space $\mathfrak{H}$ and satisfying the following conditions: I. The spectrum of the operator T lies on the unit circle. II. The operator $H = I - T^*T$ is completely continuous. III. There exists a chain belonging to the operator T and separating its spectrum*.

Under these assumptions, I. Ts. Gokhberg and M. G. Krein established ([1]) the existence of such a maximal chain of orthoprojectors $\mathfrak{P} = \{P\}$ that

$$T = U(I + V), \qquad U = \int_{\mathfrak{P}} \exp(i\varphi(P))\, dP,$$

where $\varphi(P)$ ($P \in \mathfrak{P},\ 0 \leq \varphi(P) \leq 2\pi$) is a nondecreasing, left-continuous scalar function, and V is a Volterra operator for which the chain $\mathfrak{P}$ is proper.

Consider the characteristic function (see, for example, ([4]))

$$\theta_T(\lambda) = \left[-T + \lambda(I - TT^*)^{1/2}(I - \lambda T^*)^{-1}(I - T^*T)^{1/2}\right]/\mathfrak{K}_T \qquad (|\lambda| < 1) \quad (1)$$

of the contraction T, acting from the space $\mathfrak{K}_T = \overline{(I - T^*T)\mathfrak{H}}$ into the space $\mathfrak{K}_{T^*} = \overline{(I - TT^*)\mathfrak{H}}$.

In the article it was shown that under conditions I, II, III there is the multiplicative representation

$$T^*\theta_T(\lambda) = -\left(\int_{\mathfrak{P}}^{\leftarrow} \left(I + \frac{H^{1/2} dP (I - PHP)^{-2} H^{1/2}}{\lambda e^{-i\varphi(P)} - 1}\right)\right)/\mathfrak{K}_T. \qquad (2)$$

By formal transformations, justified only for the case when $H = I - T^*T$ is a nuclear operator, equality (2) was reduced to the form

$$\theta_T(\lambda) = U_0 \left(\int_{\mathfrak{P}}^{\leftarrow} \left(I - \frac{1}{2} \frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda} R^*(P) H^{1/2} \, dP \, H^{1/2} R(P) \right) \right) / \mathfrak{K}_T, \quad (3)$$

where U_0 is an isometric operator mapping the whole space $\mathfrak{K}_T$ onto the whole space $\mathfrak{K}_{T^*}$, and

$$R(P) = \int_0^P \left(I + \frac{1}{2} H^{1/2} (I - PHP)^{-1} \, dP \, H^{1/2} \right). \quad (4)$$

I. Ts. Gokhberg and M. G. Krein expressed the conjecture that representation (3) of the function $\theta_T(\lambda)$ is valid also in the general case. In the present note the validity of this conjecture is proved and, along the way, are derived

* In the present note the terminology of article ([1]) is used (see also ([2],[3])).

simpler and more convenient for investigation multiplicative representations of the characteristic functions of contraction operators. The proof of these facts is based on propositions concerning the properties of additive and multiplicative integrals over chains of orthoprojectors and the relations between them, which are possibly of independent interest.

1. The integrals of the form encountered below,

$$\int_{\mathfrak{P}} F(P) \, dP \, G(P), \qquad \int_{\mathfrak{P}}^{\sim} (I + F(P) \, dP \, G(P)), \qquad \int_{\mathfrak{P}}^{\sim} \exp(F(P) \, dP \, G(P)),$$

where $F(P)$ and $G(P)$ are operator-functions, are understood as limits in the uniform norm, in the sense of S. O. Shatunovskii, respectively of sums and products of the form

$$\sum_{j=1}^{n} F(Q_j) \, \Delta P_{jG}(Q_j), \qquad \prod_{j=1}^{n} (I + F(Q_j) \, \Delta P_{jG}(Q_j)), \qquad \prod_{i=1}^{n} \exp(F(Q_j) \, \Delta P_{jG}(Q_j)) \quad (5)$$

$$(0 = p_0 < p_1 < \ldots < p_n = I; \quad P_{j-1} \leqslant Q_j \leqslant P_j; \quad P_j, Q_j \in \mathfrak{P}; \quad \Delta P_j = P_j - P_{j-1}).$$

The integrals

$$\int_{\mathfrak{P}}^{\frown} (I + F(P)\, dP\, G(P)), \qquad \int_{\mathfrak{P}}^{\frown} \exp(F(P)\, dP\, G(P))$$

are defined analogously.*

The following assertion holds, generalizing the results of article ([5]).

Theorem 1. Let the values of the functions $X(P)$ and $Y(P)$ $(P \in \mathfrak{P})$ be linear completely continuous operators, let $\alpha(P)$ $(P \in \mathfrak{P})$ be a bounded scalar function $\left(\inf_{P\in\mathfrak{P}} |\alpha(P)| > 0\right)$, and let $X(P) = \alpha(P)X_1(P)$.

If the integrals

$$A = \int_{\mathfrak{P}} dP\, X(P), \qquad B = \int_{\mathfrak{P}} Y(P)\, dP, \qquad C = \int_{\mathfrak{P}} dP\, X(P) \int_0^P Y(Q)\, dQ, \qquad \int_{\mathfrak{P}} \alpha(P)\, dP, \tag{6}$$

exist, then the function $W(P) = I + BP(I - C)^{-1}A$ is the unique solution of the equation

$$W(P) = I + \int_0^P Y(Q)\, dQ\, X(Q)\, W(Q) \tag{7}$$

and is representable in the form

$$W(P) = \int_0^{P\,\frown} (I + B\, dQ\, A) = \int_0^{P\,\frown} (I + \alpha(Q) B\, dQ\, A_1) \left(A_1 = \int_{\mathfrak{P}} dP\, X_1(P) \right).$$

Theorem 2. If the functions $X(P)$ and $Y(P)$ satisfy the conditions of Theorem 1 and the sums of the form $\sum_{j>i} \Delta P_{jX}(Q_j)Y(Q_i)\Delta P_i$ $(P_j \leqslant Q_j \leqslant P_j)$ converge in the sense of S. O. Shatunovskii in the uniform norm to the integral

$$\int_{\mathfrak{P}} dP\, X(P) \int_0^P Y(Q)\, dQ,$$

then

$$\int_0^{P\,\frown} (I + B(Q)\, dQ\, A) = \int_0^{P\,\frown} I + Y(Q)\, dQ X(Q)). \tag{8}$$

* Let us note that in article ([1]) the integrals (2), (3), (4) were understood as limits of products of the form (5) under the condition $Q_i = P_j$.

Theorem 3. Let the values of the functions $X(P), Y(P)$ be such that, for any break (P^-, P^+) of the chain $\mathfrak{P}$, the equality

$$(P^+ - P^-)X(P^+)Y(P^+)(P^+ - P^-) = (P^+ - P^-)X(P^-)Y(P^-)(P^+ - P^-) = 0. \tag{9}$$

holds.

If $X(P) = RX'(P)$, $Y(P) = Y'(P)R^*$,

$$\sup_{P\in\mathfrak{P}} \|X'(P)\| < \infty, \qquad \sup_{P\in\mathfrak{P}} \|Y'(P)\| < \infty,$$

where R belongs to the Hilbert-Schmidt class, and the integral

$$\int_0^P (I + Y(Q)\, dQ\, X(Q))$$

exists, then

$$\int_0^P (I + Y(Q)\, dQ\, X(Q)) = \int_0^P \exp(Y(Q)\, dQ\, X(Q)).$$

2. All multiplicative integrals that will be obtained below are to be understood as limits of products of the form (5) under the condition

$$P_{j-1} < Q_j \leq P_j \qquad (j = 1, 2, \ldots, n).$$

*

Theorem 4. If the contraction operator T satisfies conditions I, II, III, then

$$\theta_T(\lambda) = U_0 \int_{\mathfrak{P}} \left(I - \frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda}\, dF(P) \right) / \mathfrak{K}_T =$$

$$= U_0 \int_{\mathfrak{P}} \left(I - \frac{1}{2}\, \frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda}\, R^*(P) H^{1/2}\, dP\, H^{1/2} R(P) \right) / \mathfrak{K}_T,$$

where

$$U_0 = (T^*)^{-1} \left(H^{1/2}(2I + V^*)^{-1} H^{1/2} - I \right) / \mathfrak{K}_T = -T \int_{\mathfrak{P}} \Big(I + \frac{1}{2}(I -$$

$$- H^{1/2} P H^{1/2})^{-1} H^{1/2}\, dP\, H^{1/2} \Big) / \mathfrak{K}_T$$

is an isometric operator mapping the whole space

$$\mathfrak{K}_T = \overline{(I - T^*)\mathfrak{H}}$$

onto the whole space

$$\mathfrak{K}_{T^*} = \overline{(I - TT^*)\mathfrak{H}},$$

$$F(P) = 2H^{1/2}(2I + V)^{-1}P(2I + V^*)^{-1}H^{1/2} = \frac{1}{2}\int_0^P R^*(Q)H^{1/2}\,dQ\,H^{1/2}R(Q),$$

$$R(P) = I + H^{1/2}(I + V)^{-1}P(2I + V)^{*-1}H^{1/2} =$$

$$= \int_0^P \left(I + \frac{1}{2}H^{1/2}(I + V)^{-1}\,dQ\,H^{1/2}\right)\int_0^P \left(I + \frac{1}{2}(I - H^{1/2}QH^{1/2})^{-1}H^{1/2}\,dQ\,H^{1/2}\right).$$

If $H = I - T^*T$ is a nuclear operator, then **

$$\theta_T(\lambda) = U_0 \int_{\mathfrak{P}} \exp\left(-\frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda}\,dF(P)\right)/\mathfrak{K}_T =$$

$$= U_0 \int_{\mathfrak{P}} \exp\left(-\frac{1}{2}\,\frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda}\,R^*(P)H^{1/2}\,dP\,H^{1/2}R(P)\right)/\mathfrak{K}_T; \qquad (11)$$

$$R(P) = \int_0^P \exp\left(\frac{1}{2}(I - H^{1/2}QH^{1/2})^{-1}H^{1/2}\,dQ\,H^{1/2}\right). \qquad (12)$$

* If the function $\varphi(P)$ and the chain P are continuous, then all multiplicative integrals occurring in this item may be understood as limits of products of the form (5).

** The second of formulas (11) and formula (12) were first obtained by I. Ts. Gokhberg and M. G. Kreĭn ([1]).

3. Let the function $\theta(\lambda)$ ($|\lambda| < 1$), whose values are linear bounded operators in $\mathfrak{H}$, satisfy the conditions: 1) it is holomorphic and invertible inside the unit disk; 2) $\|\theta(\lambda)\| \le 1$ ($|\lambda| < 1$); 3) $\|\theta(0)f\| < \|f\|$ for all $f \neq 0$. Then, as was shown in ([4]), there exists a contraction T, satisfying conditions I and II, for which $\theta(\lambda) = U_*\theta_T(\lambda)U$, where the operators U and U_* isometrically map the spaces $\mathfrak{H}$ and $\mathfrak{R}_{T^*}$, respectively, onto $\mathfrak{R}_T$ and $\mathfrak{H}$. If, moreover: 4) there exists a point λ_0 ($|\lambda_0| < 1$) such that $I - \theta^*(\lambda_0)\theta(\lambda_0) \in \mathfrak{S}_\omega$, then (1) $I - T^*T \in \mathfrak{S}_\omega$ and, in view of the results of V. I. Macaev ([8]), the operator T has a chain separating its spectrum. This circumstance, together with Theorem 4, leads to a generalization of the results of ([9–12]) on the multiplicative representation of operator-functions analytic in the unit disk in the direction indicated in the note ([1]).

The results of the present note extend to the case where $\|T\| \ge 1$. In addition, if the operator $H = I - T^*T$ belongs to the symmetrically normed ideal $\mathfrak{S}$, then the integrals (10), (11), and (12) converge in the norm of $\mathfrak{S}$.

The author expresses his sincere gratitude to I. Ts. Gohberg for assistance in carrying out the work.

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Received
17 III 1966

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Note: Figure translations are in progress. See original paper for figures.