ON OPEN MAPPINGS

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Abstract

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I. In ([1]) P. S. Aleksandrov showed that countably multiple open mappings of compacta do not increase their dimension, and in ([2]) A. N. Kolmogorov showed that the set of points of local topologicity of a countably multiple open mapping of a compactum X is everywhere dense in X. P. S. Aleksandrov raised the question of extending his theorem to arbitrary bicompacta. Below we give generalizations of the theorems of Aleksandrov and Kolmogorov; in particular, an answer is given to P. S. Aleksandrov' s question.

All spaces under consideration are assumed to be completely regular, and all mappings are continuous mappings "onto." By a complete space is meant a space complete in the sense of Čech, i.e., a space that is a set of type G_δ in some (any) of its bicompact extensions. A space is locally complete if each of its points has a neighborhood whose closure (and hence the neighborhood itself) is complete.

By a σ-discrete set we mean a set that decomposes into the sum of a countable number of discrete (i.e., also closed) subsets. Obviously, every σ-discrete bicompactum (and even every finally compact space) is at most countable. A mapping $f : X \to Y$ will be called σ-discrete if every set $f^{-1}(y)$, $y \in Y$, is σ-discrete. The mapping f is locally σ-discrete if for each point $x \in X$ there exists a neighborhood on which the mapping f is σ-discrete. Examples of σ-discrete mappings are countably multiple mappings.

We now formulate the first general result, which generalizes A. N. Kolmogorov' s theorem.

Theorem 1. If the mapping $f : X \to Y$ is open and locally σ-discrete, and the space X is locally complete, then the set of points of local topologicity of the mapping f is everywhere dense (and open) in X.

Corollary 1. The set of points of local topologicity of an open countably multiple mapping of a complete (for example, locally bicompact) space X is everywhere dense and open in X.*

Corollary 2. The set of points of local topologicity of an open countably multiple mapping of a bicompactum X is everywhere dense in X.

We now formulate the second general result, which generalizes P. S. Aleksandrov' s theorem.

Theorem 2. If a mapping f of a locally complete normal space X onto a weakly paracompact normal space Y is open and σ-discrete, then

$$\dim Y \leq \operatorname{loc}\dim X \leq \dim X^{**}.$$

Corollary 3. If a mapping f of a complete normal space X onto a weakly paracompact normal space Y is open

* After obtaining this result, V. Proizvolov generalized it somewhat, though only for the case of locally bicompact spaces ([8]).
** $\dim X$ is defined by means of open finite coverings. For $\operatorname{loc}\dim X$ and $\operatorname{loc}\operatorname{Ind} X$, see ([3]).

and countably many, then

$$\dim Y \leq \dim X.$$

Corollary 4. If the mapping $f : X \to Y$ of a bicompactum is open and countably many, then

$$\dim Y = \dim X.$$

Corollary 5. If the mapping $f : X \to Y$ of a complete paracompactum X is open-closed and σ-discrete, then

$$\dim Y = \dim X.$$

It follows from Corollary 1 that A. D. Taimanov' s theorem ([4]) on the non-increase of dimension under countably many open mappings of complete spaces with a countable base follows, as does the later theorem of A. V. Arhangel' skii on the non-increase of the dimension of complete metric spaces under the same mappings onto metric spaces ([5]).

The following theorem also generalizes a theorem of P. S. Aleksandrov.

Theorem 3. If the mapping f of a locally complete normal space X onto a totally normal ([3]) weakly paracompact (for example, hereditarily paracompact, or weakly paracompact perfectly normal, or metric) space Y is open and locally σ-discrete, then

$$\operatorname{Ind} X \geq \operatorname{Ind} Y.$$

Corollary 6. If the mapping f of a locally complete hereditarily paracompact space X onto a space Y is open-closed and locally σ-discrete, then

$$\operatorname{Ind} X = \operatorname{Ind} Y.$$

Corollary 7. If the mapping f of a bicompactum X onto a bicompactum Y is open and countably many, and at least one of the bicompacta X or Y is perfectly normal, then

$$\operatorname{Ind} X = \operatorname{Ind} Y.$$

Theorem 3 can be somewhat generalized by replacing in it the totally normal space Y by a Dowker space: a hereditarily normal space will be called **Dowker** if every open subset of X has a point-finite open covering by sets of type F_σ in X. Examples of Dowker spaces are hereditarily normal hereditarily weakly paracompact spaces. Dowker spaces behave, with respect to the dimensions Ind and dim, in the same way as totally normal spaces.

Theorem 4. If the space X is Dowker and $A \subseteq X$, then

$$\dim A \leq \dim X, \qquad \operatorname{Ind} A \leq \operatorname{Ind} X.$$

Theorem 5. If for a Dowker space X one of the conditions is fulfilled: a) $X = A \cup B$, $\operatorname{Ind} A \leq n$, $\operatorname{Ind} B \leq n$, and the set A is closed in X; b) $X = \bigcup_{i=1}^{\infty} X_i$, $\operatorname{Ind} X_i \leq n$, and the sets X_i are closed in X, then

$$\operatorname{Ind} X \leq n.$$

Theorem 6. For a weakly paracompact Dowker space the relation

$$\operatorname{Ind} X = \operatorname{loc} \operatorname{Ind} X$$

holds.

Let us also note that the following is true.

Proposition 1. Let a normal subspace A of a normal space X be such that there exists a point-finite covering of the set A by sets open in A and of type F_σ in X; then

$$\dim A \leq \dim X.$$

II. In ([6]) Hausdorff proved that, under an open mapping f of a metric space X onto a metric space Y, the completeness of X implies the completeness of Y. In ([7]) E. Michael showed that Hausdorff' s theorem remains valid if Y is only paracompact. Moreover, he showed that in X there exists a set which is mapped perfectly by means of f onto Y (thus, Y nevertheless turns out to be metrizable). The following theorems generalize the results of Hausdorff and Michael.

Theorem 7. *If a mapping $f : X \to Y$ of a locally complete space X is open, then in X there is a subset X' which is mapped perfectly by means of f onto an everywhere dense subset Y' of Y. The set X' is closed in $f^{-1}(Y')$ and, in the case of a complete space X, is of type G_δ in X.*

Corollary 8. *Open images of locally complete spaces are extensions of complete spaces (they contain complete everywhere dense subspaces).*

Corollary 9. *An open image of a complete metric space is an extension of a complete metric space (and possesses the first axiom of countability).*

Theorem 8. *If a mapping $f : X \to Y$ of a locally complete space X is open, then for every paracompact $Y' \subseteq Y$ there exists in X a subset X' which is mapped perfectly by means of f onto Y'. The set X' is closed in $f^{-1}(Y')$ and, in the case of a complete space X, is of type G_δ in $f^{-1}(Y')$.*

Corollary 10. a) *If a paracompact Y is an open image of a locally complete space, then Y is a complete space; b) more generally, if a space Y is an open image of a locally complete space, then every paracompact set $Y' \subseteq Y$, closed or of type G_δ in Y, is complete.*

Corollary 11. *If a space Y is an open image of a complete metric space, then every paracompact subspace Y is metrizable, and if this subspace is closed or of type G_δ in Y, then it is completely metrizable.*

This corollary, however, also follows from the results of E. Michael.

We prove Theorem 8 for the case $Y' = Y$ and part a) of Corollary 10. We shall first suppose that the space X is complete. Denote by $\bar{f}$ the extension of the mapping f to a mapping of βX onto βY (βA denotes the maximal bicompact extension of A), and denote the set $\bar{f}^{-1}(Y)$ by $\bar{X}$. The mapping $\bar{f} : \bar{X} \to Y$ is perfect, and X is the intersection of a countable system of open sets O_i, $i = 1, 2, ...$, in $\bar{X}$.

Choose for each point $y \in Y$ a point $x(y) \in f^{-1}(y)$ and its neighborhood $Vx(y)$ in X such that

$$Vx(y) \subseteq [Vx(y)]_{\bar{X}} \subseteq O_1.$$

The system v of the sets $V_y = f(Vx(y))$ is an open cover of Y (by virtue of the openness of f). Inscribe in the cover v a locally finite open cover

$$\omega = \{U_\alpha\}, \quad \alpha \in \mathfrak{A},$$

of the space Y. For each α we fix exactly one set $Vx(y) = V_\alpha$ for which $f(Vx(y)) \supseteq U_\alpha$. The system of open sets in X

$$W_\alpha = V_\alpha \cap f^{-1}(U_\alpha)$$

is locally finite in $\bar{X}$, for the locally finite cover

$$\omega^{-1} = \{\bar{f}^{-1}(U_\alpha)\}, \quad \alpha \in \mathfrak{A}$$

is locally finite in $\bar{X}$ (as the inverse image of a locally finite cover). Hence it is clear that the set

$$\bar{X}_1 = \bigcup_\alpha [W_\alpha]_{\bar{X}}$$

is closed in $\bar{X}$, i.e. the mapping $\bar{f}$ is perfect on $\bar{X}_1$. Since each set $[W_\alpha]_{\bar{X}}$ is contained in some $[Vx(y)]$, we have $\bar{X}_1 \subseteq O_1$. From the construction of the sets W_α it is also clear that the set

$$X_1 = \bigcup_\alpha W_\alpha$$

is open in X, i.e. on X_1 the mapping f is open and $f(X_1) = Y$. Replacing, at the second step, the space X, the space $\bar{X}$, and the set O_1, respectively, by the set X_1, the set $\bar{X}_1$, and the set $O_2' = O_2 \cap \bar{X}_1$, we obtain a set $X_2 \subseteq O_2'$, closed in $\bar{X}_1$, perfectly mapped onto Y by means of $\bar{f}$, and a set $X_2 \subseteq \bar{X}_2$, open in X_1, mapped by means of f onto the whole space Y. Continuing the process, we obtain sets $\bar{X}_i$, $i = 1, 2, ...$, closed in $\bar{X}$, the intersection F of which is contained in and closed-

then in $\bigcap O_i = X$. Since each $\overline{X}_i$ was mapped perfectly onto Y by means of $\overline{f}$, and since $\overline{X}_{i-1} \subseteq \overline{X}_i$, the intersection F is nonempty and, by means of $\overline{f}$, which coincides on $F \subseteq X$ with f, is mapped perfectly onto Y.

The set F is the intersection not only of the sets $\overline{X}_i$, but also of the sets X_i (for one may assume that $[X_{i+1}]_X \subseteq X_i$), i.e., it has type G_δ in X.

Since the perfect image of a complete space is complete, the space Y is complete (for the set F is complete).

If now the space X is locally complete, then it is the open image of a complete space Z under a mapping g. In Z, by what has been proved, there will be found a closed set F, mapped perfectly onto Y by means of $f \cdot g$. Thus the space Y is complete and part a) of Corollary 10 is proved. The set $g(F) = X'$ is closed in X and is mapped perfectly onto Y. Theorem 8 is proved (for $Y' = Y$).

Note added in proof. The result of part a) of Corollary 10 was recently also obtained by Wicke.

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Note: Figure translations are in progress. See original paper for figures.